Math 341: Sign of a permutation

In this note we will define the sign of a permutation. This will be useful for problems 4-8 of HW2.

Let n be a positive integer. A permutation $\pi \in S_n$ is called a *transposition* if it is of the form $\pi = (a_1\ a_2)$, i.e. π consists of a single cycle, and this cycle contains just two numbers.

Now let $\pi \in S_n$ be an arbitrary permutation. We will prove in lecture that π can always be written as a product of transpositions, i.e. $\pi = \pi_1\pi_2\cdots\pi_k$, where $\pi_1, \ldots, \pi_k \in S_n$ are transpositions. We define

$$\mathrm{sgn}(\pi) = (-1)^k,$$

i.e. $\mathrm{sgn}(\pi) = 1$ if k is even and $\mathrm{sgn}(\pi) = -1$ if k is odd. It's not immediately obvious that $\mathrm{sgn}(\pi)$ is well defined. Indeed, π can be written as a product of transpositions in many different ways, i.e. we could write $\pi = \pi_1\pi_2\cdots\pi_k$ or $\pi = \pi_1'\pi_2'\cdots\pi_{k'}'$, where $\pi_1, \ldots, \pi_k$ are transpositions, and $\pi_1', \ldots \pi_{k'}'$ are transpositions. It need not be the case that $k = k'$, but it *is* is the case that $(-1)^k = (-1)^{k'}$, i.e. k is even if and only if k' is even.