Math 341 Homework 5

- Due Thursday, March 29 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- You are allowed (and encouraged) to use results proved in class on your homework.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1. Prove that for each positive integer n, there are about $\sqrt{8n/3}$ pentagonal numbers less than n. More precisely, prove that for every $\epsilon > 0$, there is a number N so that for all $n \geq N$, the number of pentagonal numbers less than n is at most $(1+\epsilon)\sqrt{8n/3}$ and is at least $(1-\epsilon)\sqrt{8n/3}$.

Remark: we could write this as "the number of pentagonal numbers less than n is $\sqrt{8n/3}\big(1+o(1)\big)$."

2 a. Let $a_1, a_2, ..., a_n$ be a sequence of real numbers, with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Prove that

$$\sum_{k=1}^{n}(-1)^{k-1}a_k \leq a_1.$$

b. Prove that for $n \geq 5$, $p(n) \leq F_{n+1}$, where $p(n)$ is the partition function of n and F_n is the n–th Fibonacci number (recall that $F_1 = 1,\ F_2 = 1$, and $F_3 = 2$). Hint: part a and induction might be helpful.

3. To state this problem, we need to introduce some notation. Let n be a positive integer and let λ be a partition of n. Write $\lambda = 1^{a_{1,\lambda}}2^{a_{2,\lambda}}\cdots n^{a_{n,\lambda}}$.

With this notation, prove that for each positive integer n,

$$\sum_{\lambda \text{ a partition of n}} \Big(\prod_{i=1}^{n} i^{a_{i,\lambda}} a_{i,\lambda}!\Big)^{-1} = 1.$$

4. Prove that for $k > n/2$, the number of permutations in S_n that have a cycle of length k is $n!/k$. Question 3 might be helpful.

5. Three candidates, A, B, and C, are running in an election. Suppose that $3n$ voters secretly cast their votes by writing one candidate's name on a ballot, and then putting it into a ballot box. The ballot only contains the candidate's name; no other information. The result is a three-way tie: each candidate receives exactly n votes.

The votes are counted as follows: an election official takes ballots out of the box one by one, and keeps a running tally of how many votes each candidate has received at that point. We know before any votes are counted, each candidate has received 0 votes, and after all $3n$ votes are counted, each candidate has received n votes.

How many different ways can the ballots be removed from the box so that at every point, the number of votes for candidate A is at least as large as the number of votes for candidate B, and the number of votes for candidate B is at least as large as the number of votes for candidate C? Hint: politicians and clowns aren't so different.

6. How many Young tableau are associated to each of the following partitions?

a. $\lambda = 1^3 2^2 4^1$.

b. $\lambda = 3^2 4^2 5^1$.

c. $\lambda = 2^1 3^3 4^1$.

7. For each of $n = 1, 2, 3, 4$, write down all of the partitions of n, all of the Young Tableau associated to each partition, and compute the quantity

$$\sum_{\lambda} f_{\lambda}^2,$$

where the sum is taken over all partitions λ of n (recall that f_λ is an integer; f_λ^2 means the square of this integer).

Remark: there appears to be a pattern here. Do you have a conjecture for what happens for larger values of n?