Math 341 Homework 3 Solutions

Fibonacci numbers

1. Prove that if n is divisible by three, then F_n is even; i.e. F_3, F_6, F_9, etc. are even.

Solution. We will prove the result by induction. First, observe that $F_3 = 2$ is even. Now suppose that k is a positive integer and that F_{3k} is even. We have $F_{3(k+1)} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k}$. Since $2F_{3k+1}$ is even and F_{3k} is even, and the sum of two even numbers is even, we conclude that $F_{3(k+1)}$ is even. Thus F_{3k} is even for every positive integer k. (Note that $F_0 = F_{3\cdot 0}$ is also even).

2. Prove that if $n \geq 6$ and n is even, then F_n is composite (i.e. it is not prime). Hint: try expanding out the formula $F_n = F_{n-1} + F_{n-2}$ multiple times.

Solution. Fix a positive integer $n \geq 3$. First, we will prove by induction that for every positive integer $k \leq n-1$,, $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$. When $k = 1$, we have $F_n = F_{n-1} + F_{n-2} = F_2F_{n-1} + F_1F_{n-2}$, so the base case holds. Now suppose the result has been proved for some $k \leq n-2$. Then

$$\begin{aligned}
F_n &= F_{k+1}F_{n-k} + F_kF_{n-k-1} \\
&= F_{k+1}(F_{n-k-1} + F_{n-k-2}) + F_kF_{n-k-1} \\
&= (F_{k+1} + F_k)F_{n-k-1} + F_{k+1}F_{n-k-2} \\
&= F_{k+2}F_{n-k-1} + F_{k+1}F_{n-k-2} \\
&= F_{(k+1)+1}F_{n-(k+1)} + F_{(k+1)}F_{n-(k+1)-1},
\end{aligned}$$

which completes the induction step. Now suppose $n \geq 6$ is even, i.e. $n = 2m$ for some positive integer m. Applying the above result with $k = m = n/2$, we have

$$F_n = F_{m+1}F_m + F_mF_{m-1} = F_m(F_{m+1} + F_{m-1}).$$

Since $n \geq 6$, $m \geq 3$ and thus $F_m \geq 2$ and $(F_{m+1} + F_{m-1}) \geq 2$, i.e. F_m can be written as the product of two integers, each of which is ≥ 2. Thus F_n is composite.

Generating functions

3. Expand $\frac{2t}{1-8t+15t^2}$ as a power series (i.e. write it in the form $\sum_{n\geq 0} a_n t^n$, and compute the numbers a_n).

Solution. We have $1-8t+15t^2 = (1-5t)(1-3t)$. Thus we can compute a partial fraction expansion:

$$\frac{2t}{1-8t+15t^2} = \frac{A}{1-5t} + \frac{B}{1-3t}.$$

Solving $A(1-3t) + B(1-5t) = 2t$, we see that $A = 1$ and $B = -1$, i.e.

$$\frac{2t}{1-8t+15t^2} = \frac{1}{1-5t} - \frac{1}{1-3t}.$$

Thus the power series expansion for $\frac{2t}{1-8t+15t^2}$ is

$$\sum_{n=0}^{\infty} 5^n t^n - \sum_{n=0}^{\infty} 3^n t^n = \sum_{n=0}^{\infty} (5^n - 3^n) t^n.$$

4. Consider the sequence a_n defined by $a_0 = 0$ and $a_{n+1} = 3a_n + 2$ for $n \geq 0$. Using the method of generating functions, write down a formula for a_n.

Solution. Define $\phi(t) = \sum_{n=0}^{\infty} a_n t^n$. We have

$$\begin{aligned}
\phi(t) &= \sum_{n=1}^{\infty} a_n t^n \\
&= \sum_{n=1}^{\infty} (3a_{n-1} + 2) t^n \\
&= 3t \sum_{n=0}^{\infty} a_n t^n + 2 \sum_{n=0}^{\infty} t^n - 2 \\
&= 3t\phi(t) + \frac{2}{1-t} - 2,
\end{aligned}$$

so $\phi(t)(1-3t) = \frac{2}{1-t} - 2$, i.e. $\phi(t) = \frac{2}{(1-t)(1-3t)} - \frac{2}{1-3t}$. Taking a partial fraction expansion of

$$\frac{2}{(1-t)(1-3t)} = \frac{-1}{1-t} + \frac{3}{1-3t},$$

we obtain

$$\begin{aligned}
\phi(t) &= \frac{-1}{1-t} + \frac{3}{1-3t} - \frac{2}{1-3t} \\
&= \frac{-1}{1-t} + \frac{1}{1-3t}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (3^n - 1) t^n$$

Thus $a_n = 3^n - 1$ for each integer $n \geq 0$.

5. Consider the sequence a_n defined by $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 0$. Using the method of generating functions, write down a formula for a_n.

Solution. Define $\phi(t) = \sum_{n=0}^{\infty} a_n t^n$. We have

$$\begin{aligned}
\phi(t) &= \sum_{n=1}^{\infty} a_n t^n \\
&= t + \sum_{n=2}^{\infty} a_n t^n \\
&= t + \sum_{n=2}^{\infty} (2a_{n-1} - a_{n-2}) t^n \\
&= t + 2t \sum_{n=0}^{\infty} a_n t^n - t^2 \sum_{n=0}^{\infty} a_n t^n \\
&= t + 2t\phi(t) - t^2\phi(t),
\end{aligned}$$

so $\phi(t)(1-2t+t^2) = t$, or

$$\phi(t) = \frac{t}{1-2t+t^2} = \frac{t}{(1-t)^2}.$$

Since the denominator has a repeated root, we can't use the method of partial fractions to write $\frac{t}{(1-t)^2} = \frac{A}{1-t} + \frac{B}{1-t}$. However, we can compute directly the series expansion of $\frac{t}{(1-t)^2}$. Lets take a few derivatives: $\phi'(t) = \frac{1+t}{(1-t)^3}$, $\phi''(t) = \frac{2(2+t)}{(1-t)^4}$, $\phi'''(t) = \frac{6(3+t)}{(1-t)^5}$. We'll prove by induction that $\phi^{(k)}(t) = \frac{k!(k+t)}{(1-t)^{k+2}}$. We've already established the base case $k = 1$ (and also $k = 2$ and $k = 3$ for that matter). Now for the induction step:

$$\begin{aligned}
\phi^{(k+1)}(t) &= \Big(\frac{k!(k+t)}{(1-t)^{k+2}}\Big)' \\
&= k! \frac{(k+t)(-(k+2)(1-t)^{k+1}) - (1-t)^{k+2}}{(1-t)^{2k+4}} \\
&= k! \frac{(k+1)\big(1-t\big)^{k+2}\big((k+1)+t\big)}{(1-t)^{2k+4}} \\
&= (k+1)! \frac{(k+1)+t}{(1-t)^{(k+1)+2}},
\end{aligned}$$

which completes the induction. Thus the function $\phi(t)$ has the expansion $\phi(t) = \sum_{n=0}^{\infty} b_n t^n$, where $b_n = \frac{1}{n!}\phi^{(n)}(0) = \frac{1}{n!}\frac{n!(n+0)}{(1-0)^{n+2}} = n$. We conclude that $a_n = n$ for each $n \geq 0$.

We can also verify this directly, since $n + 2 = 2(n + 1) - n$. (perhaps the moral of the story is that sometimes having a lucky guess is easier than using generating functions).

Permutations

6. Let $\pi \in S_n$. Prove the formula $\mathrm{sgn}(\pi) = (-1)^{n-C(\pi)-F(\pi)}$ from lecture.

Solution. Recall that if $(a_1\ a_2\ \ldots\ a_k)$ is a cycle, then we can write

$$(a_1\ a_2\ \ldots\ a_k) = (a_1\ a_2)(a_1\ a_3)(a_1\ a_4)\cdots(a_1\ a_k).$$

The right hand side of the above equation is a product of $k-1$ transpositions, and thus its sign is $(-1)^{k-1}$. Now suppose $\pi \in S_n$, and suppose π can be written as a product of disjoint cycles of lengths $k_1, k_2, \ldots, k_t$ $t = C(\pi)$. We have $n = k_1 + k_2 + \ldots + k_t + F(\pi)$, and

$$\begin{aligned}
\mathrm{sgn}(\pi) &= (-1)^{k_1-1}(-1)^{k_2-1}\cdots(-1)^{k_t-1} \\
&= (-1)^{k_1+k_1+\ldots+k_t}(-1)^{-t} \\
&= (-1)^{n-F(\pi)}(-1)-t \\
&= (-1)^{n-C(\pi)-F(\pi)}.
\end{aligned}$$

7. a. Let $\pi \in S_n$ be of the form $\pi = (a_1\ a_2\ \ldots\ a_k)$. Prove that π is even if and only if k is odd. For part b, we will call k the "length" of the cycle.

Solution. As noted in problem 6, we can write $\pi = (a_1\ a_2\ \ldots\ a_k) = (a_1\ a_2)(a_1\ a_3)(a_1\ a_4)\cdots(a_1\ a_k)$. This is a product of $k-1$ transpositions. Thus π is even if and only if k is odd.

b. Let $\pi \in S_n$, and consider a representation of π as a product of disjoint cycles. Prove that π is even if and only if there are an even number of even length cycles (and any number of odd length cycles).

Solution. Write π as a product of disjoint cycles of lengths $k_1, k_2, \ldots, k_t$, $t = C(\pi)$. Then $\operatorname{sgn}(\pi) = (-1)^{k_1-1}(-1)^{k_2-1} \cdots (-1)^{k_t-1}$. If k_j is odd then $k_j - 1$ is even, so $(-1)^{k_j-1} = 1$. Conversely, if k_j is even, then $k_j - 1$ is odd, so $(-1)^{k_j-1} = -1$. Thus $\operatorname{sgn}(\pi) = -1$ if and only if there are an odd number of cycles of even length.