Math 341 Homework 1

- Due Thursday, January 25 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- You are allowed (and encouraged) to use results proved in class on your homework.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Sets

For these problems, we will need some notation not discussed in class. If S and T are sets, $S \cup T$ is the set containing all elements that are in S or in T (or in both). This is called the *union* of S and T. Similarly, $S \cap T$ is the set containing all elements that are in S and in T. This is called the *intersection* of S and T.

We can extend this definition to take a union of multiple sets. For example, if $S_1, \dots S_k$ are sets, we write $S_1 \cup S_2 \cup \dots \cup S_k$, or $\bigcup_{i=1}^k S_i$ to denote the union of $S_1, \dots, S_k$; this is the set of elements that are contained in at least one of the sets $S_1, \dots, S_k$. Similarly, we write $S_1 \cap S_2 \cap \dots \cap S_k = \bigcap_{i=1}^k S_i$ to denote the intersection of $S_1, \dots, S_k$; this is the set of elements that are contained in all of the sets $S_1, \dots, S_k$.

1. Let S and T be sets of finite cardinality (i.e. both $|S|$ and $|T|$ are finite). Prove that

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

2. Let S be a nonempty set. Without using the binomial theorem, prove that S has the same number of subsets of even and of odd cardinality. Hint: it might be useful to consider the cases where $|S|$ is even and where $|S|$ is odd separately.

3. Let k and n be positive integers. Let $A_1, \dots A_k$ be sets. Suppose that

$$\left|\bigcup_{i=1}^{k} A_i\right| = n^2,$$

that

$$|A_i| \geq 2n \text{ for every } 1 \leq i \leq k,$$

and that

$$|A_i \cap A_j| \leq 1 \quad \text{for every } 1 \leq i < j \leq k.$$

Prove that $k \leq n$.

Binomial theorem

4. Let p be a prime number. Prove that there exists a polynomial $Q(x)$ with positive integer coefficients so that

$$(1+x)^p - (1+x^p) = pQ(x).$$

5. Prove that for every positive integer n and every integer k, we have

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Hint: there's a reason this is is in the "Binomial Theorem" section.

The size of $n!$

6. Recall that a polynomial is a function of the form $P(n) = a_C n^C + a_{C-1} n^{C-1} + \ldots + a_0$, where $C \geq 0$ is an integer and $a_0, \ldots, a_C$ are real numbers.

Prove that for every polynomial P, there exists a number N so that $n! > P(n)$ for all $n \geq N$.

7. Prove that for every real number $C > 1$, there exists a number N so that $n! > C^n$ for all $n \geq N$.

Statistical physics and entropy

8. In this problem we will investigate the entropy of two different gases in the same box using the lattice model discussed in class.

Consider the system consisting of k_1 oxygen molecules and k_2 nitrogen molecules in a box. The box contains n possible locations where a gas molecule can reside, and at most one molecule can be located in each location. All oxygen molecules are indistinguishable, and all nitrogen molecules are indistinguishable, but oxygen and nitrogen molecules can be distinguished from each other.

What is the entropy of this system? Prove that your answer is correct.

9. Consider a system consisting of k oxygen molecules in a box. The box contains n possible locations. Will adding an additional oxygen molecule to the box increase the entropy of the system? Prove that your answer is correct. (Hint: the answer may depend on the value of k and n).