1. (10 points) Let 
\[ f(x) = x^2 + (1 - x) \sin \left( \frac{1}{1 - x} \right). \]

Using the \( \epsilon - \delta \) definition of a limit, prove that
\[ \lim_{x \to 1} f(x) = 1. \]

Note: For this problem, you cannot use the sum rule, product rule, etc. for limits; I want you to prove things “by hand.”

**Solution.**
First, note that \( D(f) = \mathbb{R} \setminus \{1\} \), so the domain requirement is met. Next, let \( \epsilon > 0 \). Select \( \delta = \min(\epsilon/4, 1) \). Then if \( 0 < |x - 1| < \delta \), we have

\[
|f(x) - 1| = |x^2 + (1 - x) \sin \left( \frac{1}{1 - x} \right) - 1| \\
\leq |x^2 - 1| + |(1 - x) \sin \left( \frac{1}{1 - x} \right)| \\
= |x - 1||x + 1| + |1 - x| |\sin \left( \frac{1}{1 - x} \right)| \\
< \delta |x + 1| + \delta |\sin \left( \frac{1}{1 - x} \right)| \\
\leq 3\delta + \delta \\
= 4\delta \\
= \epsilon. 
\]

On the second line we used the triangle inequality. On the third line we used the fact that \( |ab| = |a| |b| \). On the fourth line we used the fact that \( |x - 1| < \delta \). On the fifth line we used the fact that \( |x + 1| \leq 3 \) (since \( |x - 1| < \delta \) and \( \delta \leq 1 \)) and \( |\sin \left( \frac{1}{1 - x} \right)| \leq 1. \)
2. (10 points) Let $a, b, c, d$ be real numbers, with $a < b$ and $c < d$. Prove that if $[a, b] \cup [c, d]$ is a closed interval, then $[a, b] \cap [c, d]$ is not the empty set, i.e. $[a, b] \cap [c, d] \neq \emptyset$.

**Solution.**

First, let's suppose $a \leq c$. We will show that if $b < c$ then $[a, b] \cup [c, d]$ is not a closed interval. We will do a proof by contradiction. Recall that a closed interval is a set of the form $[e, f]$ with $e < f$; or a set of the form $(-\infty, f]$; or a set of the form $[e, \infty)$. Since $a - 1$ is not an element of $[a, b] \cup [c, d]$, $[a, b] \cup [c, d]$ certainly cannot be an interval of the form $(-\infty, f]$. Similarly, since $\max(b, d) + 1$ is not an element of $[a, b] \cup [c, d]$, $[a, b] \cup [c, d]$ certainly cannot be an interval of the form $[e, \infty)$. Thus if $[a, b] \cup [c, d]$ is a closed interval, it must be of the form $[e, f]$ for some $e < f$. Since $b \in [a, b] \cup [c, d]$, we must have $e \leq b$. Similarly, since $c \in [a, b] \cup [c, d]$, we must have $f \geq c$. But since $(b + c)/2$ is not in $[a, b] \cup [c, d]$, we must have that $(b + c)/2$ is not in $[e, f]$, and thus either $b < (b + c)/2 < e \leq b$ or $c > (b + c)/2 > f \geq c$; this is impossible. Thus if $b < c$ then $[a, b] \cup [c, d]$ is not a closed interval.

This implies that if $[a, b] \cup [c, d]$ is a closed interval, then $b \geq c$. If $b \leq d$ then $b \in [a, b]$ and $b \in [c, d]$, so $b \in [a, b] \cap [c, d]$ and thus $[a, b] \cap [c, d]$ is non-empty. On the other hand, if $b > d$ then $d \in [a, b]$ and $d \in [c, d]$, so $d \in [a, b] \cap [c, d]$ and thus $[a, b] \cap [c, d]$ is non-empty. In either case, $[a, b] \cap [c, d]$ is non-empty.

Finally, if $c < a$ then the above argument remains true if we interchange the roles of $[a, b]$ and $[c, d]$.

**Remark.** Students are also allowed to prove the result by drawing a picture (or series of pictures), but they must explain why their picture is correct.
3. (10 points) For this problem, you may use the fact that for every real number $a$, $\lim_{x \to a} \sin(x) = \sin(a)$ and $\lim_{x \to a} \cos(x) = \cos(a)$. Let

$$f(x) = \frac{2\sin(x) - x\cos(x) + 10x^3 + 2x + 1}{x^3 - 1}.$$ 

Compute

$$\lim_{x \to 2} f(x),$$

and prove that your answer is correct. For this problem, you are allowed (and encouraged) to use all of the limit rules discussed in class.

**Solution.**

Using the rule $\lim_{x \to 2} x = 2$ and the product rule (twice), we have $\lim_{x \to 2} x^3 = 8$. Using the limit rule $\lim_{x \to 2} 1 = 1$ and the difference rule, we have $\lim_{x \to 2} x^3 - 1 = 7$.

A similar application of the limit rule $\lim_{x \to 2} x = 2$ and the product rule gives $\lim_{x \to 2} 10x^3 = 80$ and $\lim_{x \to 2} 2x = 4$. Using the sum rule (twice), we conclude that $\lim_{x \to 2} 10x^3 + 2x + 1 = 85$.

Using the product rule and the fact that $\lim_{x \to 2} \sin(x) = \sin(2)$, we have $\lim_{x \to 2} 2\sin(x) = 2\sin(2)$.

Using the product rule, the limit rule $\lim_{x \to a} x = a$, and the fact that $\lim_{x \to 2} \cos(x) = \cos(2)$, we have $\lim_{x \to 2} x\cos(x) = 2\cos(2)$.

Thus by the sum rule, $\lim_{x \to 2} \left(2\sin(x) - x\cos(x) + 10x^3 + 2x + 1\right) = 2\sin(2) - 2\cos(2) + 85$, and by the quotient rule (which is applicable since $\lim_{x \to 2} (x^3 - 1) \neq 0$, we have $\lim_{x \to 2} f(x) = \frac{85 + 2\sin(2) - 2\cos(2)}{7}$.
4. (10 points) Determine which of the following statements are true and which are false. Write T or F beside each statement. You do not need to justify your answers.

a) \( \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} \) such that \( y^2 < x \).  F

b) \( \forall \epsilon \in \mathbb{R}, \ \epsilon > 0 \ \exists \delta \in \mathbb{R}, \ \delta > 0 \) such that \( \forall x \in \mathbb{R}, \) with \( 0 < |x - 1| < \delta \), we have \( |x^2 - 1| < \epsilon \).  T

c) \( \exists x \in \mathbb{R} \) such that \( \forall y \in [-1, 1], \) we have \( x > y \).  T

d) \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{Z}, \) we have \( x > y \).  F

e) \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{R}, \) we have \( xy > y^2 \).  F