Midterm 2: Question 4

To solve this problem, we know that the derivative of a constant function is 0, but we haven't yet proved that if the derivative of a function is always 0, then it is constant. So we need to both prove that the derivative is zero, and that this implies it is constant.

Part I: Prove that $\forall\ y \in \mathbb{R},\ f'(y) = 0$

- Statement: $\lim_{x\to y} \frac{f(x)-f(y)}{x-y} = 0$
- Proof: We will use an epsilon-delta proof for this problem.
  - Domain requirement is met as $D(f) = \mathbb{R}$
  - Let $\varepsilon > 0$, choose $\delta = \varepsilon$.
  - Let $x \in \mathbb{R}$ with $0 < |x - y| < \delta$
    * To prove that $f'(x) = 0$, we need $|\frac{f(x)-f(y)}{x-y} - 0| < \varepsilon$
  - We know $|f(x) - f(y)| \leq |x - y|^2$
  - As $|x - y| > 0$, $\frac{|f(x)-f(y)|}{|x-y|} \leq |x - y|$
  - Then $|\frac{f(x)-f(y)}{x-y}| \leq |x - y|$
  - So $|\frac{f(x)-f(y)}{x-y} - 0| \leq |x - y| < \delta = \varepsilon$
  - Therefore, $\forall\ y \in \mathbb{R}$, the derivative $f'(y) = 0$

Part II: Prove that if $f'(x) = 0$, $f(x) = C$, with $C \in \mathbb{R}$

- Proof: We will use the Mean Value Theorem.
  - As $\forall\ y \in \mathbb{R},\ f'(y) = 0$, f is differentiable on $\mathbb{R}$
  - Then $\forall\ a, b, \in \mathbb{R}$, with $a < b$, $[a, b] \subset D(f)$, f is continuous on $[a, b]$, and f is differentiable on (a, b)
  - Then by the Mean Value Theorem, $\exists\ c \in (a, b)$s.t. $\frac{f(b)-f(a)}{b-a} = f'(c) = 0$
  - $f(b) - f(a) = 0(b - a) = 0$
  - $f(b) = f(a)\ \forall\ a, b \in \mathbb{R}$, so $f(x)$ is constant, so $f(x) = C$, $C \in \mathbb{R}$