Math 120 Midterm 1 Practice 2 Solutions

1. (10 points) Let
$$f(x) = x^2 + (1-x)\sin\big(\frac{1}{1-x}\big).$$
Using the ϵ-δ definition of a limit, prove that
$$\lim_{x\to 1} f(x) = 1.$$
Note: For this problem, you cannot use the sum rule, product rule, etc. for limits; I want you to prove things "by hand."

Solution.
First, note that $D(f) = \mathbb{R}\backslash\{1\}$, so the domain requirement is met. Next, let $\epsilon > 0$. Select $\delta = \min(\epsilon/4,\ 1)$. Then if $0 < |x-1| < \delta$, we have

$$\begin{aligned}
|f(x) - 1| &= \big|x^2 + (1-x)\sin(\frac{1}{1-x}) - 1\big| \\
&\leq |x^2 - 1| + \big|(1-x)\sin(\frac{1}{1-x})\big| \\
&= |x-1|\ |x+1| + |1-x|\ \big|\sin(\frac{1}{1-x})\big| \\
&< \delta|x+1| + \delta\big|\sin(\frac{1}{1-x})\big| \\
&\leq 3\delta + \delta \\
&= 4\delta \\
&= \epsilon.
\end{aligned} \tag{1}$$

On the second line we used the triangle inequality. On the third line we used the fact that $|ab| = |a|\ |b|$. On the fourth line we used the fact that $|x-1| < \delta$. On the fifth line we used the fact that $|x+1| \leq 3$ (since $|x-1| < \delta$ and $\delta \leq 1$) and $\big|\sin(\frac{1}{1-x})\big| \leq 1$.

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2. (10 points) Let a, b, c, d be real numbers, with $a < b$ and $c < d$. Prove that if $[a, b] \cup [c, d]$ is a closed interval, then $[a, b] \cap [c, d]$ is not the empty set, i.e. $[a, b] \cap [c, d] \neq \emptyset$.

Solution.
First, lets suppose $a \leq c$.

We will show that if $b < c$ then $[a, b] \cup [c, d]$ is not a closed interval. We will do a proof by contradiction. Recall that a closed interval is a set of the form $[e, f]$ with $e < f$; a set of the form $(-\infty, f]$; or a set of the form $[e, \infty)$. Since $a - 1$ is not an element of $[a, b] \cup [c, d]$, $[a, b] \cup [c, d]$ certainly cannot be an interval of the form $(-\infty, f]$. Similarly, since $\max(b, d) + 1$ is not an element of $[a, b] \cup [c, d]$, $[a, b] \cup [c, d]$ certainly cannot be an interval of the form $[e, \infty)$. Thus if $[a, b] \cup [c, d]$ is a closed interval, it must be of the form $[e, f]$ for some $e < f$. Since $b \in [a, b] \cup [c, d]$, we must have $e \leq b$. Similarly, since $c \in [a, b] \cup [c, d]$, we must have $f \geq c$. But since $(b + c)/2$ is not in $[a, b] \cup [c, d]$, we must have that $(b + c)/2$ is not in $[e, f]$, and thus either $b < (b + c)/2 < e \leq b$ or $c > (b + c)/2 > f \geq c$; this is impossible. Thus if $b < c$ then $[a, b] \cup [c, d]$ is not a closed interval.

This implies that if $[a, b] \cup [c, d]$ is a closed interval, then $b \geq c$. If $b \leq d$ then $b \in [a, b]$ and $b \in [c, d]$, so $b \in [a, b] \cap [c, d]$ and thus $[a, b] \cap [c, d]$ is non-empty. On the other hand, if $b > d$ then $d \in [a, b]$ and $d \in [c, d]$, so $d \in [a, b] \cap [c, d]$ and thus $[a, b] \cap [c, d]$ is non-empty. In either case, $[a, b] \cap [c, d]$ is non-empty.

Finally, if $c < a$ then the above argument remains true if we interchange the roles of $[a, b]$ and $[c, d]$.

Remark. students are also allowed to prove the result by drawing a picture (or series of pictures), but they must explain why their picture is correct.

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3. (10 points) For this problem, you may use the fact that for every real number a, $\lim_{x\to a}\sin(x) = \sin(a)$ and $\lim_{x\to a}\cos(x) = \cos(a)$. Let

$$f(x) = \frac{2\sin(x) - x\cos(x) + 10x^3 + 2x + 1}{x^3 - 1}.$$

Compute

$$\lim_{x\to 2} f(x),$$

and prove that your answer is correct. For this problem, you are allowed (and encouraged) to use all of the limit rules discussed in class.

Solution.

Using the rule $\lim_{x\to 2} x = 2$ and the product rule (twice), we have $\lim_{x\to 2} x^3 = 8$. Using the limit rule $\lim_{x\to 2} 1 = 1$ and the difference rule, we have $\lim_{x\to 2} x^3 - 1 = 7$.

A similar application of the limit rule $\lim_{x\to 2} x = 2$ and the product rule gives $\lim_{x\to 2} 10x^3 = 80$ and $\lim_{x\to 2} 2x = 4$. Using the sum rule (twice), we conclude that $\lim_{x\to 2} 10x^3 + 2x + 1 = 85$.

Using the product rule and the fact that $\lim_{x\to 2}\sin(x) = \sin(2)$, we have $\lim_{x\to 2} 2\sin(x) = 2\sin(2)$.

Using the product rule, the limit rule $\lim_{x\to a} x = a$, and the fact that $\lim_{x\to 2}\cos(x) = \cos(2)$, we have $\lim_{x\to 2} x\cos(x) = 2\cos(2)$.

Thus by the sum rule, $\lim_{x\to 2}\left(2\sin(x) - x\cos(x) + 10x^3 + 2x + 1\right) = 2\sin(2) - 2\cos(2) + 85$, and by the quotient rule (which is applicable since $\lim_{x\to 2}(x^3 - 1) \neq 0$, we have $\lim_{x\to 2} f(x) = \frac{85+2\sin(2)-2\cos(2)}{7}$.

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4. (10 points) Determine which of the following statements are true and which are false. Write T or F beside each statement. You do not need to justify your answers.

a) $\forall\ x \in \mathbb{R}\ \exists\ y \in \mathbb{R}$ such that $y^2 < x$. F

b) $\forall \epsilon \in \mathbb{R},\ \epsilon > 0\ \exists\ \delta \in \mathbb{R},\ \delta > 0$ such that $\forall\ x \in \mathbb{R},$ with $0 < |x-1| < \delta,$ we have $|x^2 - 1| < \epsilon$. T

c) $\exists\ x \in \mathbb{R}$ such that $\forall\ y \in [-1,1],$ we have $x > y$. T

d) $\exists\ x \in \mathbb{R}$ such that $\forall\ y \in \mathbb{Z},$ we have $x > y$. F

e) $\exists\ x \in \mathbb{R}$ such that $\forall\ y \in \mathbb{R},$ we have $xy > y^2$. F