Math 120 Midterm 1 Practice 1 Solutions

1. (10 points) Let f be a function whose domain is $\mathbb{R}$. Suppose that for all $x \in (-1, 1)$,

$$x^2 \leq f(x) \leq |x|.$$

Prove that $\lim_{x\to 0} f(x) = 0$.

Solution. By the limit rule "$\lim_{x\to a} x = a$", we have $\lim_{x\to 0} x = 0$. By the product rule, we have $\lim_{x\to 0} x^2 = (\lim_{x\to 0} x)^2 = 0$. Thus $\lim_{x\to 0} x = \lim_{x\to 0} x^2 = 0$. Thus by the squeeze theorem applied to the functions x, f, and x^2 on the domain $(-1, 1)$, we have that $\lim_{x\to 0} f(x) = 0$.

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2. (10 points) Let $S \subset \mathbb{R}$ be a set. Using the quantifiers $\exists$ and $\forall$, write the sentences a) "S has an upper bound," b) "S has a least upper bound," and c) "S does not have a least upper bound."

Solution.
a) $\exists y \in \mathbb{R}$ s.t. $\forall\ x \in \S,\ x \leq y$

b) $\exists y \in \mathbb{R}$ s.t. $\forall\ x \in \S,\ x \leq y,$ and $\forall t > 0,\ \exists z \in S$ s.t. $z > y - t$.

c) $\forall y \in \mathbb{R}, \big(\exists x \in S$ s.t.$x > y$ or $\exists\ t > 0$ s.t. $\forall x \in S,\ x \leq y - t\big)$

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3. (10 points) Let $f(x) = x^2 + x + 1$. Using the definition of a limit, prove that

$$\lim_{x \to 4} f(x) = 21.$$

Remember that there are two conditions you must establish in order to prove that a limit exists.

Solution. First, note that $D(f) = \mathbb{R}$. Thus there certainly exists a number $t > 0$ so that $\{x \in \mathbb{R} \colon 0 < |x - 4| < t\} \subset D(f)$. In fact, any number $t > 0$ will suffice.

Now, let $\epsilon > 0$. Select $\delta = \min(\epsilon/10, 1)$. Then if $0 < |x - 4| < \delta$, then since $\delta \leq 1$, we have $3 \leq 4 - \delta < x < 4 + \delta \leq 5$, so $|x| \leq 5$. Thus by repeated use of the triangle inequality,

$$\begin{aligned}
|f(x) - 21| &= |x^2 + x + 1 - 21| \\
&\leq |x^2 - 16| + |x - 4| + |1 - 1| \\
&\leq |(x - 4)(x + 4)| + |x - 4| \\
&\leq |x - 4||x + 4| + |x - 4| \\
&\leq |x - 4|(|x| + 4) + |x - 4| \\
&< \delta(5 + 4) + \delta \\
&= 10\delta \\
&\leq \epsilon.
\end{aligned}$$

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4. (10 points) $f(x) = \cos(x) + 0.1\sin(x) + x$. (Remember that in this class, angles are measured in radians). Prove that the statement

$$\lim_{x\to 1} f(x) = 100$$

is false.

Solution. First, note that $D(f) = \mathbb{R}$, so the domain requirement is met. Next, observe that for every $x \in \mathbb{R}$, $|\cos(x)| \leq 1$ and $|\sin(x)| \leq 1$, and thus $|0.1\sin(x)| \leq 0.1$.

Select $\epsilon = 1$, and let $\delta > 0$. If $\delta > 1$, select $x = 2$. Otherwise, select $x = 1 + \delta/2$. In either case, we have $0 < |x - 1| < \delta$, and $1 < x \leq 2$. Then

$$\begin{aligned}
|f(x) - 100| &= |100 - f(x)| \\
&\geq 100 - |f(x)| \\
&= 100 - |\cos(x) + 0.1\sin(x) + x| \\
&\geq 100 - |\cos(x)| - 0.1|\sin(x)| - |x| \\
&\geq 100 - 1 - 0.1 - 2 \\
&> 96 \\
&> \epsilon.
\end{aligned}$$

We conclude that the statement $\lim_{x\to 1} f(x) = 100$ is false.