Math 120 Practice problem Solutions

1. Use the $\epsilon - \delta$ definition of the limit to prove that

\[ \lim_{x \to 1} x^3 + 2x + 1 = 4. \]

Proof. First, the domain of the function $x^3 + 2x + 1$ is $\mathbb{R}$, so the domain requirement is met. Let $\epsilon > 0$. Select $\delta = \min(1, \epsilon/9)$. Then if $0 < |x - 1| < \delta$, we have that $|x| \leq 2$, and thus

\[
|x^3 + 2x + 1 - 4| \leq |x^3 - 1| + |2x - 2| \\
\leq |(x - 1)(x^2 + x + 1)| + 2|x - 1| \\
= |x - 1| |x^2 + x + 1| + 2|x - 1| \\
\leq 7|x - 1| + 2|x - 1| \\
\leq 9|x - 1| < \epsilon.
\]

2. Use the $\epsilon - \delta$ definition of the limit to prove that

\[ \lim_{x \to 0} e^{x^2} = 1. \]

Solution. First, the domain of the function $e^{x^2}$ is $\mathbb{R}$, so the domain requirement is met. Let $\epsilon > 0$. Select $\delta = \min(1, \epsilon/5)$. Observe that if $|x - 0| < \delta$, then $|x^2| < |x| < \delta$. We proved in lecture that $|e^{|x|} - 1| < 5|x|$ if $|x| < 1$. Thus if $|x^2| < 1$, we have $|e^{x^2} - 1| < 5|x^2| < 5|x| < 5\delta < \epsilon$.

3. Define

\[ f(x) = \begin{cases} 
  x, & x \in \mathbb{Q}, \\
  0, & x \notin \mathbb{Q}
\end{cases} \]

Prove that for every $x \in \mathbb{R}$, $f$ is not differentiable at $x$.

Solution. First, we can use the quotient property for limits to see that $f(x)$ is not continuous if $x \neq 0$, and thus certainly $f(x)$ is not differentiable if $x \neq 0$. It remains to prove that $f(x)$ is not differentiable at $x = 0$. Suppose $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = L$, i.e. $\lim_{h \to 0} f(h)/h = L$. If $h \neq 0$, $f(h)/h = 1$ if $h \in \mathbb{Q}$ and 0 if $h \notin \mathbb{Q}$. We proved in lecture that this function does not have a limit at any point in $\mathbb{R}$, and thus the function $g(x) = 1, x \in \mathbb{Q}; 0, x \neq \mathbb{Q}$ does not have a limit as $x \to 0$. By the limits are a “local property” theorem, we conclude that the statement “$\lim_{h \to 0} f(h)/h = L$” is false for every real number $L$. Thus $f$ is not differentiable at any point $x \in \mathbb{R}$.

4. a. Compute the zeroth, first, and second term (plus error term) of the Taylor expansion of $\arcsin(x)$ around the point $c = 0$; i.e. when you apply Taylor’s theorem, you should have $n = 3$.

b. Use part a to prove that $\arcsin(1/2) \leq 1/2 + 1/\sqrt{3}$.

Solution. a. We have

\[ \arcsin(x) = x + \frac{1}{6} \left( \frac{3x^2}{(1-x^2)^{5/2}} + \frac{1}{(1-x^2)^{3/2}} \right)x^3, \]
where $x_1$ lies in the interval between 0 and $x$.

b. Observe that if $0 < x_1 < 1/2$, then $\frac{3x_1^2}{(1-x_1)^{3/2}} \leq \frac{8}{3\sqrt{3}}$, and $\frac{1}{(1-x^2)^{3/2}} \leq \frac{8}{3\sqrt{3}}$. Thus if $x = 1/2$, we have

$$\frac{1}{6} \left( \frac{3x_1^2}{(1-x_1)^{3/2}} + \frac{1}{(1-x^2)^{3/2}} \right) x^3 \leq \frac{1}{9\sqrt{3}}.$$ 

Thus $\arcsin(1/2) \leq \frac{1}{2} + \frac{1}{9\sqrt{3}}$.

5. Evaluate

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^4}.$$ 

Justify any steps that you make.

**Solution.** Let $f(x) = \frac{e^{-1/x^2}}{x^4}$. We have $D(f) = \mathbb{R} \setminus \{0\}$, so the domain requirement is met. First, observe that since $x^4 > 0$ for $x \neq 0$, and $e^y > 0$ for all $y$, we have that $\frac{e^{-1/x^2}}{x^4} > 0$ for all $x \neq 0$. Let $h(x) = 0$. Next, note that if $x > 0$, $e^x > 1 + x + x^2/2! + x^3/3! > x^3/3!$, so if $x \neq 0$, then $e^{1/x^2}x^4 > (1/x^2)^{3}x^4 > 3! > x^{-2}/3!$. Thus $f(x) = 1/e^{1/x^2}x^4 < 6x^2$. Let $h(x) = 6x^2$; we have $g(x) \leq f(x) \leq h(x)$ for all $x \neq 0$. Since $\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = 0$, by the squeeze theorem we have $\lim_{x \to 0} f(x) = 0$, as desired.

6. Let $f$ be a function that is increasing on $[0,1]$. Prove that for every $c \in (0,1]$, $\lim_{x \to c^-} f(x)$ exists (and is a real number).

**Solution.** Let $S = \{f(x): x \in (0,c)\}$. Since $f(x) \leq f(1)$ for all $x \in [0,1]$, $S$ has an upper bound (namely $f(1)$), and thus has a least upper bound. Call this number $L$. Observe that since $L$ is the least upper bound for $S$, for every $\epsilon > 0$, there exists $y \in S$ with $y > L - \epsilon$. Since $y \in S$, there exists $x_1 \in (0,1]$ with $f(x_1) = y$. Note that we must have $x_1 < c$, since otherwise this would violate the property that $f(x)$ is increasing. Thus if we define $\delta = c - x_1$, we have $\delta > 0$. If $0 < c - x < \delta$, then $|f(x) - L| = L - f(x) < L - f(y) < L - (L - \epsilon) = \epsilon$. Thus $\lim_{x \to c^-} f(x) = L$, and in particular, the limit exists.

7. Let $f(x) = \sin(\log(-x))$ and let $S = \{x \in D(f): f(x) = 0\}$. Does $S$ have a least upper bound? If so, find it (and prove that your answer is correct). If not, prove that $S$ does not have a least upper bound.

**Solution.** Yes: First, note that $D(f) = (-\infty,0)$. Thus $S \subset D(f) = (-\infty,0)$ so $S$ has an upper bound (namely 0), so $S$ must have a least upper bound. We will prove that this least upper bound is 0. Since 0 is an upper bound, the least upper bound cannot be larger than 0. Now, let $\epsilon > 0$, and suppose $-\epsilon$ was an upper bound for $S$. This would imply that for all $x \in (-\epsilon,0)$, $f(x) \neq 0$, i.e. for all $x \in (-\epsilon,0)$, $\log(-x)$ is not a multiple of $\pi$. However, if $\epsilon > 0$ then we can select an integer $n > 0$ so that $0 < e^{-n\pi} < \epsilon$; but then $\log(-x) = -n\pi$, so $\sin(\log(-x)) = 0$.

We conclude that $\sup(S) = 0$.

8. Let $f(x)$ be a function that is differentiable at every point $x \in \mathbb{R}$ and whose range is a subset of $(0,\infty)$. Prove that the curves $y = f(x)$ and $y = \log(f(x))$ have horizontal tangent lines at exactly the same values of $x$.

**Solution.** Since $\text{range}(f) \subset (0,\infty)$, $D(\log f) = \mathbb{R}$. Suppose the curve $y = f(x)$ has a horizontal tangent line at $x = c$. Then $f'(c) = 0$. This means that $\frac{d}{dx}(\log f(x))|_c = f'(c)/f(c) = 0$, so the curve $y = \log(x)$ also has a horizontal tangent line at $c$. Conversely, if $y = \log f(x)$ has a horizontal tangent line at $c$ then $f'(c)/f(c) = 0$, so $f'(c) = 0$ and thus $y = f(x)$ has a horizontal tangent line at $c$. 

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9. Let $f(x)$ and $g(x)$ be functions that are two-times differentiable for all $x \in \mathbb{R}$, and suppose $f(x)g(x) = 1$ for all $x \in \mathbb{R}$. Show that the following relation holds for all $x \in \mathbb{R}$ where the denominators are non-zero

$$\frac{f''(x)}{f'(x)} - 2 \frac{f'(x)}{f(x)} - \frac{g''(x)}{g'(x)} = 0.$$  

Solution. First, $f'(x)/f(x) + g'(x)/g(x) = 0$. Differentiate this again to get $f''(x)/f(x) - (f'(x))^2/(f(x))^2 + g''(x)/g(x) - (g'(x))^2/(g(x))^2 = 0$. Multiply by $f(x)/f'(x) = -g(x)/g'(x)$ and re-arrange.

10. Prove that if $F_1$ and $F_2$ are anti-derivatives for the function $f$ on the interval $(a, b)$, then $F_1(x) - F_2(x)$ is a constant.

Solution. If $F_1$ and $F_2$ are anti-derivatives for the same function $f$ on $(a, b)$, then for all $x \in (a, b)$, $F_1'(x) - F_2'(x) = f(x) - f(x) = 0$. Let $x_0 \in (a, b)$ and let $A = F_1(x) - F_2(x)$. Applying the mean value theorem to the differentiable function $F_1 - F_2$, we have that for all points $x \in (a, b)$ with $x \neq x_0$,

$$(F_1(x) - F_2(x)) - A = (F_1 - F_2)(x) - (F_1 - F_2)(x_0) = (F_1 - F_2)'(c)(x - x_0) = 0(x - x_0) = 0,$$

where $c$ is a point on the interval between $x_0$ and $x$. Thus for all points $x \in (a, b)$ with $x \neq x_0$, we have $F_1(x) - F_2(x) = A$. Of course we also have $F_1(x_0) - F_2(x_0) = A$. We conclude that $F_1 - F_2 = A$.

### Differential Equations

Since we haven’t had any homework problems on differential equations, here are a bunch to help you study.

11. Carbon dating is a method used to determine how long ago an organism died. There are three isotopes of carbon that are common on earth: carbon-12 and carbon-13 (which have half-lives of $\infty$ years), and carbon-14 (which has a half-life of 5730 years). Most of the carbon on earth is carbon-12, but some is carbon-14 (we’ll forget about carbon-13 for this problem; it might as well be carbon-12). When an organism is alive, it absorbs carbon-12, carbon-13, and carbon-14 equally easily, so the ratio of carbon-12 to carbon-14 is the same inside all living organisms. For simplicity, we will assume that this ratio has also been the same at all points in the past. Once an organism dies, however, it stops absorbing new carbon, so the carbon-14 inside the organism decays into carbon-12. If a fossil has $\%3$ as much carbon-14 as a modern living organism does, how old is the fossil?

Solution. The amount of carbon-14 in a sample can be modeled using the equation $y(t) = C_0 2^{-t/\lambda} = C_0 e^{-(\log 2/\lambda)t}$, where $t$ is the time (in years), $C_0$ is the initial amount of carbon-14 in the sample and $\lambda = 5730$ is the half-life of carbon-14. We have $y(t) = 0.03$, and must solve for $t$, i.e. $0.03 = e^{-(\log 2/5730)t}$, or $t = \frac{\log(0.03)}{-(\log 2/5730)} = 28988$ years.

12. Solve the initial value problem

$$y'(x) + xy(x) = 0,$$

$$y(0) = 1.$$
13. Solve the initial value problem
\[
y'(x) = \sin(x),
y(0) = 17.
\]

*Solution.* Let \( y(x) = -\cos(x) + 18 \) then \( y(0) = -\cos(0) + 18 = -1 + 18 = 17 \), and \( y'(x) = \sin(x) \), as desired.

14. Solve the initial value problem
\[
\begin{align*}
y'(x) - 3y(x) &= e^{2x}, \\
y(0) &= 0.
\end{align*}
\]

What is the largest interval containing 0 in which the solution is defined?

*Solution.* This is a first order linear ordinary differential equation of the form \( y'(x) + P(x)y(x) = Q(x) \) with \( P(x) = -3 \) and \( Q(x) = e^{2x} \). Observe that \( A(x) = -3x \) is an anti-derivative for \( P(x) \) with \( P(0) = 0 \), and \( B(x) = -e^{-x} \) is an anti-derivative of \( e^{A(x)}Q(x) = e^{-3x}e^{2x} = e^{-x} \). Thus the solution to the differential equation is \( y(x) = Ce^{3x} + e^{3x}(-e^{-x}) = Ce^{3x} - e^{2x} \). Since \( y(0) = 0 \), we have \( C = 1 \), i.e. the solution to the initial value problem is
\[
y(x) = e^{3x} - e^{2x}.
\]

This function is defined for all \( x \in \mathbb{R} \), so the largest interval for which the solution is defined is \( \mathbb{R} \).

15. Solve the initial value problem
\[
\begin{align*}
y'(x) - xy(x) &= x^3, \\
y(0) &= 0.
\end{align*}
\]

What is the largest interval containing 0 in which the solution is defined?

*Solution.* This is a first order linear ordinary differential equation of the form \( y'(x) + P(x)y(x) = Q(x) \) with \( P(x) = -x \) and \( Q(x) = x^3 \). Observe that \( A(x) = -x^2/2 \) is an anti-derivative for \( P(x) \) with \( P(0) = 0 \), and \( B(x) = -e^{-x^2/2}(2 + x^2) - 2 \) is an anti-derivative of \( e^{A(x)}Q(x) = e^{-x^2/2}x^3 \) with \( B(0) = 0 \).

Thus the solution to the differential equation is
\[
y(x) = Ce^{x^2/2} + e^{x^2/2}( -e^{-x^2/2}(2 + x^2) - 2) = (C - 2)e^{x^2/2} - x^2 - 2.
\]

Since \( y(0) = 0 \), we have \( C = 4 \), so the solution to the initial value problem is
\[
y(x) = 2e^{x^2/2} - x^2 - 2e.
\]

Since this function is defined for all \( x \in \mathbb{R} \), the largest interval containing 0 on which the solution is defined in \( \mathbb{R} \).

16. Consider the initial value problem
\[
\begin{align*}
y'(x) &= y(x)^{1/3}, \\
y(0) &= 0.
\end{align*}
\]

Prove that \( y_1(x) = 0 \), \( y_2(x) = \left(\frac{2}{3}x\right)^{3/2} \) and \( y_3(x) = -\left(\frac{2}{3}x\right)^{3/2} \) are all solutions to the initial value problem. This should surprise you.
Remark This example shows that the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = C_0$ might not have a unique solution if the function $f$ is “badly behaved”. In this example, the function $f(x, y) = y^{1/3}$ is not differentiable with respect to $y$.

Solution. First, we can immediately see that $y_1(0) = y_2(0) = y_3(0) = 0$, so each of $y_1, y_2,$ and $y_3$ satisfy the initial condition. Next, we have $y_1'(x) = 0 = y_1^{1/3}$, so $y_1$ solves the initial value problem. We also have $y_2'(x) = \frac{3}{2} \left( \frac{2}{3} x \right)^{1/2} \left( \frac{2}{3} \right) = \left( \frac{2}{3} x \right)^{1/2} y_2^{1/3}$, and similarly $y_3'(x) = -\frac{3}{2} \left( \frac{2}{3} x \right)^{1/2} \left( \frac{2}{3} \right) = -\left( \frac{2}{3} x \right)^{1/2} = -y_2^{1/3}$.