Math 120 Homework 9 Solutions

L’Hopital’s rule

1. In this problem we will use Taylor’s theorem to understand the indeterminate form 0/0. Let \( f \) and \( g \) be functions that are twice differentiable on the interval \([-1, 1]\), and suppose that \( f'' \) and \( g'' \) are continuous on \([-1, 1]\). Suppose that \( f(0) = 0 \), \( g(0) = 0 \), and \( g'(0) \neq 0 \).

(a) Prove that for each \( x \in (0, 1) \) for which \( g(x) \neq 0 \), there exist numbers \( x_1, x_2 \in (0, x) \) so that

\[
\frac{f(x)}{g(x)} = \frac{f'(0)x + \frac{f''(x_1)}{2}x^2}{g'(0)x + \frac{g''(x_2)}{2}x^2}.
\]

(b) Prove that

\[
\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.
\]

Solution

a. Apply Taylor’s theorem with \( n = 2 \) to \( f \) and \( g \), with \( c = 0 \). We conclude that there are points \( 0 \leq x_1, x_2 \leq x \) so that

\[
f(x) = f(0) + f'(0)x + \frac{f''(x_1)}{2}x^2 = f'(0)x + \frac{f''(x_1)}{2}x^2,
\]

and \( g(x) = g'(0)x + \frac{g''(x_2)}{2}x^2 \).

Since \( g(x) \neq 0 \), we have

\[
\frac{f(x)}{g(x)} = \frac{f'(0)x + \frac{f''(x_1)}{2}x^2}{g'(0)x + \frac{g''(x_2)}{2}x^2}.
\]

b. Since \( f''(x) \) is continuous on \([-1, 1]\), by the extreme value theorem there exists \( M \) so that \( |f''(x)| \leq M \) for all \( x \in [-1, 1] \). Similarly there exists \( N \) so that \( |f''(x)| \leq N \) for all \( x \in [-1, 1] \).

Since \( g'(0) \neq 0 \) and \( g' \) is continuous, there exists \( t > 0 \) so that \( |g'(x)| > 0 \) for all \( x \in (-t, t) \) (this follows from the \( \epsilon - \delta \) definition of continuity at 0: let \( \epsilon = |g'(0)| \) and let \( t = \delta \)). Thus for each \( x \in (0, t) \), we have

\[
\left| \frac{f(x)}{g(x)} - \frac{f'(0)}{g'(0)} \right| = \left| \frac{f''(0)\frac{x^2}{2} + f''(x_1)\frac{x^2}{2} - f''(x_2)\frac{x^2}{2}}{g'(0)(g'(0)x + \frac{g''(x_2)}{2}x^2)} \right|
\]

\[
= \frac{x^2}{2} \left| \frac{f''(x_1)g'(0) - f''(x_2)g'(0)}{g'(0)(g'(0)x + \frac{g''(x_2)}{2}x^2)} \right|
\]

\[
= \frac{Mg'(0) + Nf'(0)}{g'(0)(2g'(0) + \frac{g''(x_2)}{2}x)} \quad x
\]

\[
\leq \left| \frac{Mg'(0) + Nf'(0)}{g'(0)^2} \right| x.
\]

Let \( t_1 = \min(t, g'(0)/N) \). Then if \( x \in (0, t_1) \), we have

\[
\left| \frac{f(x)}{g(x)} - \frac{f'(0)}{g'(0)} \right| \leq \left| \frac{Mg'(0) + Nf'(0)}{(g'(0))^2} \right| x.
\]
Now, let $\varepsilon > 0$. Select
\[
\delta = \min \left( t_1, \varepsilon \frac{(g'(0))^2}{M g'(0) + N f'(0)} \right).
\]
Then for all $x \in \mathbb{R}$ with $0 < x < \delta$, we have
\[
\left| \frac{f(x)}{g(x)} - \frac{f'(0)}{g'(0)} \right| < \varepsilon.
\]
This proves that $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$, as desired.

2. Use L'Hopital’s rule to prove the following

(a) Prove that if $f$ and $g$ are polynomials with $f(x) = a_n x^n + \ldots + a_0$ and $g(x) = b_n x^n + \ldots + b_0$, with $a_n \neq 0$, $b_n \neq 0$, then
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_n}.
\]

(b) Prove that if $f$ and $g$ are polynomials with $\deg(g) > \deg(f)$, then
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.
\]

Solution

a. We will prove the result by induction on $n$. The base case $n = 0$ is trivial: we have $f(x) = a_0$ and $g(x) = a_0$, so
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{a_0}{b_0} = a_0/b_0.
\]
Next, suppose the result has been proved for all pairs of polynomials of degree at most $n$, and let $f$ and $g$ be polynomials of degree $n + 1$. Since $\deg f \geq 1$, the limit rule for quotients of polynomials (proved in lecture) says that $\lim_{x \to \infty} f(x) = \infty$ or $-\infty$. Similarly, $\lim_{x \to \infty} g(x) = \infty$ or $-\infty$. We proved in lecture that for any polynomial $g$, there exists a number $R > 0$ so that if $x > R$ then $g(x) \neq 0$. Thus $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is of the indeterminate form $\frac{\infty}{\infty}$, and L’Hopital’s rule can be applied. If $f(x) = a_{n+1} x^{n+1} + a_n x^n + \ldots + a_0$, then $f'(x) = (n+1) a_{n+1} x^n + n a_n x^{n-1} + \ldots + a_1$, and similarly if $g(x) = b_{n+1} x^{n+1} + b_n x^n + \ldots + b_0$, then $g'(x) = (n+1) b_{n+1} x^n + n b_n x^{n-1} + \ldots + b_1$. These are polynomials of degree $n$, so by the induction assumption,
\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \frac{(n+1) a_{n+1}}{(n+1) b_{n+1}} = \frac{a_{n+1}}{b_{n+1}}.
\]
Thus by L’Hopital’s rule,
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \frac{a_{n+1}}{b_{n+1}},
\]
which completes the induction step and finishes the proof.

b. We will again prove the result by induction on the degree of $f$. If $f(x) = a_0$ has degree 0, then since $g$ has degree $> 0$, we have $\lim_{x \to \infty} g(x) = \infty$, so $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{a_0}{g(x)} = 0$.

Now suppose the result has been proved for all pairs of polynomials where the numerator has degree
\[ \leq n, \text{ and let } f, g \text{ be polynomials where } \deg f = n + 1 \text{ and } \deg g > \deg(f). \text{ The same argument from part a shows that we may apply L’hopital’s rule to conclude that} \]

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0,
\]

where for the second equality we used the induction assumption, since \( \deg f' = n \) and \( \deg g' > \deg f' \). This completes the induction step and concludes the proof.

5+5 = 10 points.

3. (this problem requires material covered in Monday Nov 20) Consider \( \lim_{x \to \infty} \frac{\cosh x}{\sinh x} \). Prove that this is of the indeterminate form \( \infty/\infty \). Prove that if we repeatedly apply L’Hopital’s rule (without simplifying the expression), we will always have a limit of the indeterminate form \( \infty/\infty \).

Hint: induction might be a useful tool to prove this.

Solution. First, note that the domain of \( \sinh(x) \) and \( \cosh(x) \) is \( \mathbb{R} \). Recall \( \sinh(x) = \frac{1}{2} (e^x - e^{-x}) \).

Since \( e^x > 1 + x \) and \( e^{-x} \leq e^0 = 1 \) for all \( x \geq 0 \), we have \( \sinh(x) \geq \frac{x}{2} \) for all \( x \geq 0 \). Thus for all \( M \geq 0 \), if we select \( R = 2M \) then for all \( x > R \) we have \( \sinh(x) > M \). In particular, this implies \( \lim_{x \to \infty} \sinh(x) = \infty \). An analogous argument (using the fact that \( \cosh(x) \geq \frac{x}{2} \) for all \( x > 0 \)) proves that \( \lim_{x \to \infty} \cosh(x) = \infty \). Thus \( \lim_{x \to \infty} \frac{\cosh(x)}{\sinh(x)} \) is of the indeterminate form \( \infty/\infty \).

We will now prove that for all \( n \geq 0 \), \( \lim_{x \to \infty} \frac{\cosh(n x)}{\sinh(n x)} \) of the indeterminate form \( \infty/\infty \). Note that

\[
\sinh^{(n)}(x) = \begin{cases} 
\sinh(x), & n \text{ is even}, \\
\cosh(x), & n \text{ is odd}.
\end{cases}
\]

Similarly,

\[
\cosh^{(n)}(x) = \begin{cases} 
\cosh(x), & n \text{ is even}, \\
\sinh(x), & n \text{ is odd}.
\end{cases}
\]

Thus

\[
\lim_{x \to \infty} \frac{\cosh^{(n)}(x)}{\sinh^{(n)}(x)} = \begin{cases} 
\lim_{x \to \infty} \frac{\cosh(x)}{\sinh(x)}, & n \text{ is even}, \\
\lim_{x \to \infty} \frac{\cosh(x)}{\sinh(x)}, & n \text{ is odd}.
\end{cases}
\]

Since \( \lim_{x \to \infty} \sinh(x) = \infty \) and \( \lim_{x \to \infty} \cosh(x) = \infty \), \( \lim_{x \to \infty} \frac{\cosh^{(n)}(x)}{\sinh^{(n)}(x)} \) is always of the indeterminate form \( \infty/\infty \), so repeated applications of L’Hopital’s rule won’t help us evaluate the limit.

6 points

Logarithmic differentiation

4. (a) Let \( f \) and \( g \) be functions that are differentiable at \( c \). Suppose that \( f(c) > 0 \) and \( g(c) > 0 \). Using logarithmic differentiation, the sum rule for derivatives and the chain rule, prove that

\[
(fg)'(c) = f'(c)g(c) + f(c)g'(c).
\]

Note: you are not allowed to use the product rule to prove Equation (1).

(b) Let \( f \) and \( g \) be functions that are differentiable at \( c \). Using part a, prove that

\[
(fg)'(c) = f'(c)g(c) + f(c)g'(c).
\]

Note: you are not allowed to use the product rule to prove Equation (2).
Solution

a. Recall that since \( f \) is differentiable at \( c \), it is also continuous at \( c \). Since \( f(c) > 0 \), there exists a number \( \delta_1 > 0 \) so that if \( |x - c| < \delta_1 \) then \( |f(x)| > 0 \). Similarly, there exists a number \( \delta_2 > 0 \) so that if \( |x - c| < \delta_2 \) then \( |g(x)| > 0 \). Let \( \delta = \min(\delta_1, \delta_2) \).

Since \( f(x) > 0 \) and \( g(x) > 0 \) for all \( x \) with \( |x - c| < \delta \), we have that \( \log(f(x)) \) and \( \log(g(x)) \) are well-defined for all \( x \) with \( |x - c| < \delta \). Thus

\[
\frac{f''(c)}{f(c)} + \frac{f'(c)}{g(c)} = \frac{d}{dx}(\log(f(x)) + \log(g(x)))|_{x=c} = \frac{d}{dx} \log(fg(x))|_{x=c} = \frac{(fg)'(c)}{fg(c)}.
\]

Multiplying both sides by the non-zero quantity \( f(c)g(c) \), we obtain

\[
(fg)'(c) = f'(c)g(c) + f(c)g'(c).
\]

b. Define \( F(x) = f(x) - f(c) + 1 \), \( G(x) = g(x) - g(c) + 1 \). Then \( F \) and \( G \) are differentiable at \( c \), and \( F(c) > 0 \), \( G(c) > 0 \). Observe that

\[
F(x)G(x) = f(x)g(x) + f(c)g(x) + f(x)g(c) + f(x) + g(x).
\]

Thus by the sum rule for derivatives, we have

\[(FG)'(c) = (fg)'(c) - f'(c)g(c) - f'(c)c + f'(c) + g'(c).\] (3)

By part a, we have

\[(FG)'(c) = F'(c)G(c) + F(c)G'(c).
\]

Since \( F'(c) = f'(c) \) and \( G'(c) = g'(c) \), this becomes

\[(FG)'(c) = f'(c)(g(c) - g(c) + 1) + (f(c) - f(c) + 1)g'(c) = f'(c) + g'(c).
\]

Combining this with (3) above, we obtain

\[f'(c) + g'(c) = (fg)'(c) - f'(c)c - f'(c)c + f'(c) + g'(c),\]

so

\[(fg)'(c) = f(c)g'(c) - f'(c)g(c)\).

5+5 = 10 points