Math 120 Homework 9

- Due Friday Nov 24 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper. **Homework that is not stapled will loose 1 point.**
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

L'Hopital's rule

1. In this problem we will use Taylor's theorem to understand the indeterminate form 0/0. Let f and g be functions that are are twice differentiable on the interval $[-1,1]$, and suppose that f'' and g'' are continuous on $[-1,1]$. Suppose that $f(0)=0$, $g(0)=0$, and $g'(0)\neq 0$.

(a). Prove that for each $x\in(0,1)$ for which $g(x)\neq 0$, there exist numbers $x_1, x_2 \in (0,x)$ so that

$$\frac{f(x)}{g(x)} = \frac{f'(0)x + \frac{f''(x_1)}{2}x^2}{g'(0)x + \frac{g''(x_2)}{2}x^2}.$$

(b) Prove that

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

2. Use L'Hopital's rule to prove the following

(a) Prove that if f and g are polynomials with $f(x) = a_n x^n + \ldots + a_0$ and $g(x) = b_n x^n + \ldots + b_0$, with $a_n \neq 0,\ b_n \neq 0$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = a_n/b_n.$$

(b) Prove that if f and g are polynomials with $\deg(g) > \deg(f)$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

3. (this problem requires material covered in Monday Nov 20) Consider $\lim_{x\to\infty} \frac{\cosh x}{\sinh x}$. Prove that this is of the indeterminate form ∞/∞. Prove that if we repeatedly apply L'Hopital's rule (without simplifying the expression), we will always have a limit of the indeterminate form ∞/∞. Hint: induction might be a useful tool to prove this.

Logarithmic differentiation

4. (a) Let f and g be functions that are differentiable at c. Suppose that $f(c) > 0$ and $g(c) > 0$. Using logarithmic differentiation, the sum rule for derivatives and the chain rule, prove that

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c). \tag{1}$$

Note: you are not allowed to use the product rule to prove Equation (1).

(b) Let f and g be functions that are differentiable at c. Using part a, prove that

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c). \tag{2}$$

Note: you are not allowed to use the product rule to prove Equation (2).