Math 120 Homework 8 Solutions

Hyperbolic and inverse trigonometric functions

1. Compute the derivative of \( f(x) = \frac{\arccos x}{x^2 - 1} \). What is the domain of \( f(x) \)?

   **Solution.** This is just an exercise in using the product and chain rule. The solution is
   \[
   \frac{\sqrt{1-x^2} - 2x \arccos(x)}{(x^2 - 1)^2}
   \]
   (but you should show some work). This is well defined if the following conditions hold: \( 1 - x^2 \geq 0 \),
   i.e. \( x \in [-1, 1] \); \( x^2 - 1 \neq 0 \); \( x \in D(\arccos x) = [-1, 1] \). Thus the domain of \( f \) is \((-1, 1)\).
   (5 points)

2. Prove that \( \sinh(x+y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y) \) and \( \cosh(x+y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \).

   **Solution.** We have
   \[
   \sinh(x+y) = \frac{1}{2}(e^{x+y} - e^{-x-y}) = \frac{1}{2}(e^x e^y - e^{-x} e^{-y}) = \frac{1}{4}((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y}))
   \]
   \[
   = \sinh(x) \cosh(y) + \cosh(x) \sinh(y).
   \]
   A similar computation shows that \( \cosh(x+y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \). (you should do this computation)
   (5 points)

   **Grader’s comment:** A few people tried to be clever for the hyperbolic trigonometric functions identities, by saying \( \sinh(x) = -i \sin(ix) \) and \( \cosh(x) = \cos(ix) \) and used trigonometric identities for \( \cos \) and \( \sin \). However, \( ix \) is not a real number, so not an angle, so they cannot use the properties of the \( \cos \), \( \sin \) which they only know the definition over reals.

3. Graph the function \( f(x) = \arctan(\tan(x)) \) for all \( x \) in \( D(f) \cap [-2\pi, 2\pi] \). You do not need to prove that your graph is correct, but draw it carefully—mark your \( x \) and \( y \) axes, scale things properly, and be sure to get the domain correct.
   (5 points)

Implicit differentiation

4. At which points on the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \)? Prove that your answer is correct.

   **Solution.** The line \( y = x \) has slope 1. We need to find all point \((x, y)\) on the curve \( x^2 + 3y^2 = 1 \)
   where the curve has slope 1. If we regard \( y = y(x) \) as a function of \( x \), then we have
   \[
   0 = \frac{d}{dx}(x^2 + 3y^2) = 2x + 6y \frac{dy}{dx}
   \]
re-arranging, we obtain
\[
\frac{dy}{dx} = -\frac{2x}{6y} = -\frac{x}{3y},
\]
whenever \( y \neq 0 \). Solving \(-x/3y = 1\), \( x^2 + 3y^2 = 1 \) we obtain the single equation \( 12y^2 = 1 \), or \( y = \pm 1/\sqrt{12} \). The corresponding points are \((\sqrt{3}/2, -1/\sqrt{12})\) and \((-\sqrt{3}/2, 1/\sqrt{12})\). However, this reasoning is only valid if \( y \neq 0 \). There are two points on the curve \( x^2 + 3y^2 = 1 \) where \( y = 0 \); the points are \((1, 0)\) and \((-1, 0)\). At both these points the tangent line to the curve \( x^2 + 3y^2 = 1 \) is vertical, so it is not parallel to the line \( y = x \).

Thus the two points where the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \) are \((\sqrt{3}/2, -1/\sqrt{12})\) and \((-\sqrt{3}/2, 1/\sqrt{12})\).

(5 points)

5. Compute the slope of the tangent line of the curve \( y^5 + 2xy^3 + 3x^2y + 10x = 16 \) at the point \((1, 1)\).

Solution. If we regard \( y \) as a function of \( x \) and use implicit differentiation, then
\[
0 = \frac{d}{dx}(y^5 + 2xy^3 + 3x^2y + 10x) = 5y^4\frac{dy}{dx} + 2y^3 + 6xy^2\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} + 10.
\]

re-arranging, we get
\[
\frac{dy}{dx} = -\frac{2y^3 + 6xy + 10}{5y^4 + 6xy^2 + 3x^2}
\]

whenever the denominator is non-zero. Plugging in \((x, y) = (1, 1)\), the denominator is \( 5 + 6 + 3 = 14 \neq 0 \), so we can compute
\[
\frac{dy}{dx} = -\frac{2 + 6 + 10}{5 + 6 + 3} = -\frac{9}{7}.
\]

(5 points)