Math 120 Homework 7 Solutions

In this homework, you will construct a rather strange function.

1. Prove that if \( P(x) \) is a polynomial, then

\[
\lim_{x \to 0^+} P(1/x)e^{-\frac{1}{x}} = 0
\]

**Solution.** Recall from HW 4 \#2b that if \( g(x) = 1/x \) and if \( \lim_{x \to \infty} f \circ g(x) = L \), then \( \lim_{x \to 0^+} f(x) = L \). Let \( f(x) = P(\frac{1}{x})e^{-\frac{1}{x}} \). Then if \( x \neq 0 \), \( f \circ g(x) = P(x)e^{-x} = P(x)/e^x \). We proved in lecture that \( \lim_{x \to \infty} f \circ g(x) = 0 \), and thus \( \lim_{x \to 0^+} f(x) = 0 \).

5 points.

2. Let \( P(x) = a_nx^n + \ldots + a_0 \) be a polynomial and let \( f(x) = P(1/x) \). Prove that there exists a polynomial \( Q(x) \) so that for all \( x \neq 0 \),

\[
f'(x) = Q(1/x).
\]

**Solution.** We have

\[
f(x) = a_nx^{-n} + a_{n-1}x^{-(n-1)} + \ldots + a_1x^{-1} + a_0.
\]

We proved in class (using the quotient rule) that if \( f(x) = x^{-k} \), then \( f'(x) = -kx^{-k-1} \) for all \( x \neq 0 \). Thus using the sum rule for derivatives, we have that for all \( x \neq 0 \),

\[
f'(x) = a_n(-n)x^{-n-1} + a_{n-1}(-n-1)x^{-n} + \ldots + a_1(-1)x^{-2}.
\]

Thus if we define

\[
Q(x) = -na_nx^{n+1} - (n-1)a_{n-1}x^n - (n-2)a_{n-2}x^{n-1} - \ldots - a_1x^{-2},
\]

then \( f'(x) = Q(1/x) \) for all \( x \neq 0 \).

5 points.

3. \( P(x) = a_nx^n + \ldots + a_0 \) be a polynomial and let \( g(x) = P(1/x)e^{-1/x} \). Prove that there exists a polynomial \( R(x) \) so that for all \( x \neq 0 \),

\[
g'(x) = R(1/x)e^{-1/x}.
\]

**Solution.** Let \( P(x) = a_nx^n + \ldots + a_0 \) and let \( f(x) = P(1/x) \). Note that both \( f(x) \) and \( e^{-1/x} \) are differentiable for \( x \neq 0 \). Using the product rule, we have that

\[
g'(x) = f'(x)e^{-1/x} + f(x)(e^{-1/x})' = (f'(x) + f(x) \cdot (-1/x^2))e^{-1/x}
\]

By problem 2, we have that there exits a polynomial \( Q(x) = b_mx^m + \ldots + b_0 \) so that \( f'(x) = Q(1/x) \) for all \( x \neq 0 \). Define

\[
R(x) = Q(x) - x^2P(x) = (b_mx^m + \ldots + b_1) - (a_nx^{n+2} + a_{n-1}x^{n+1} + \ldots + a_0x^2).
\]
This is clearly a polynomial, and \( R(1/x) = f'(x) - (1/x^2)f(x) \) for all \( x \neq 0 \), so
\[
g'(x) = R(1/x)e^{-1/x}
\]
for all \( x \neq 0 \).

5 points.

4. Prove by induction that for each integer \( n \geq 1 \), there exists a polynomial \( R_n(x) \) so that if \( f(x) = e^{-1/x} \), then
\[
f^{(n)}(x) = R_n(1/x)e^{-1/x}.
\]
for all \( x \neq 0 \).

Solution.
First we will do the base case. If \( n = 1 \), then \( f^{(1)}(x) = (-1/x^2)e^{-1/x} \), so the result is true with \( R_1(x) = -x^2 \).

Next we will do the induction step. Suppose the result has been proved for some integer \( n \geq 1 \). Then for all \( x \neq 0 \),
\[
f^{(n+1)}(x) = (f^{(n)}(x))' = (R_n(1/x)e^{-1/x})' = R_{n+1}(1/x)e^{-1/x}.
\]
For the last inequality we used Problem 3 (with \( P(x) = R_n(x) \)), and we define \( R_{n+1} \) to be the output polynomial \( R \) from problem 3. This completes the induction step and thus completes the proof.

5 points.

5. Define
\[
g(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x}, & x > 0 \end{cases}
\]
a. Prove that for every number \( n \), \( g \) is \( n \)-times differentiable on \( \mathbb{R} \) (i.e. \( g^{(n)}(x) \) exists for every \( x \in \mathbb{R} \)).
b. Prove that \( g^{(n)}(0) = 0 \) for every non-negative integer \( n = 1, 2, \ldots \).

Solution.
a. Since \( g(x) = 0 \) if \( x < 0 \), by the limits are a local property rule we have that \( g^{(n)}(x) = 0 \) if \( x < 0 \). Similarly, since \( g(x) = e^{-1/x} \) if \( x > 0 \), by the limits are a local property rule and Problem 4, we have that \( g^{(n)}(x) = R_n(1/x)e^{-1/x} \) if \( x > 0 \), where \( R_n \) is the polynomial from problem 4. In particular, we know that for each integer \( n \geq 1 \), \( g \) is \( n \) times differentiable at \( x \) for all \( x \neq 0 \). In part b below, we will show that \( g \) is \( n \) times differentiable at 0.

b. We will prove by induction on \( n \) that \( g^{(n)}(0) = 0 \) for all \( n \), so in particular, \( g \) is \( n \) times differentiable at 0. We will begin with \( n = 0 \). Since \( g(x) \) is defined for all \( x \in \mathbb{R} \), \( g(x) \) is 0-times differentiable at 0, and \( g^{(0)}(0) = g(0) = 0 \). Now suppose we have shown that \( g^{(n)}(0) = 0 \). In order to prove that \( g^{(n+1)}(0) = 0 \), it suffices to prove that
\[
\lim_{x \to 0^-} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = \lim_{x \to 0^+} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = 0.
\]
Since \( g^{(n)}(0) = 0 \) and \( g^{(n)}(x) = 0 \) for all \( x < 0 \), we have
\[
\lim_{x \to 0^-} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = 0.
\]
Since \( g^{(n)}(x) = R_n(1/x)e^{-1/x} \) for all \( x > 0 \) (here \( R_n \) is the polynomial from Problem 4), we have
\[
\lim_{x \to 0^+} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = \lim_{x \to 0^+} \frac{1}{x} R_n(1/x)e^{-1/x} = 0,
\]
where for the last equality we used Problem 1. This completes the induction step and finishes the proof.

3+5 = 8 points.