Math 120 Homework 3 Solutions

Limits

1. In lecture, we discussed the limit rule: If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) with \( M \neq 0 \), then

\[
\lim_{x \to a} (f/g)(x) = L/M.
\]

However, if \( \lim_{x \to a} g(x) = 0 \), then all bets are off. Give an example of functions \( f, g \) with \( \lim_{x \to a} f(x) = 0 \), \( \lim_{x \to a} g(x) = 0 \), and:

a. \( \lim_{x \to a} (f/g)(x) = 1 \).

b. \( \lim_{x \to a} (f/g)(x) = 0 \).

c. \( \lim_{x \to a} (f/g)(x) \) does not exist (as a real number).

In each of the above problems, prove that your answer is correct.

Solution.

a. Let \( f(x) = x - a, \ g(x) = x - a \). By the difference rule for limits, \( \lim_{x \to a} f(x) = \lim_{x \to a} (x - a) = x - a \). By the limit rule “\( \lim_{x \to a} x = a \)” and by the limit rule “\( \lim_{x \to a} K = K \),” we have \( \lim_{x \to a} f(x) = a - a = 0 \). Similarly, \( \lim_{x \to a} g(x) = 0 \). Thus \( f \) and \( g \) satisfy the hypotheses of the problem.

On the other hand, \( (f/g)(x) = 1 \), if \( x \neq 0 \). Let \( h(x) = 1 \). Then \( (f/g)(x) = h(x) \) for \( x \in \mathbb{R} \setminus \{a\} \). Thus by the “limits are a local property rule,” and by the limit rule “\( \lim_{x \to a} K = K \),” we have \( \lim_{x \to a} (f/g)(x) = \lim_{x \to a} 1 = 1 \).

b. Let \( f(x) = (x - a)^2, \ g(x) = x - a \). We already proved that \( \lim_{x \to a} g(x) = 0 \). By the product rule, \( \lim_{x \to a} f(x) = (\lim_{x \to a} (x - a)) (\lim_{x \to a} (x - a)) = (0)(0) = 0 \), so \( f \) and \( g \) satisfy the hypotheses of the theorem. Now, observe that \( (f/g)(x) = (x - a) \) if \( x \neq a \). Thus by the “limits are a local property rule,” \( \lim_{x \to a} (f/g)(x) = \lim_{x \to a} (x - a) = 0 \).

c. Let \( f(x) = (x - a), \ g(x) = (x - a)^2 \). We already proved in part b that \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \) (the roles of \( f \) and \( g \) were reversed, but the proof is the same). Suppose that \( \lim_{x \to a} \frac{(x - a)^2}{(x - a)^2} = L \) for some real number \( L \). Then by the product rule for limits and part b above, \( \lim_{x \to a} \frac{(x - a)^2}{(x - a)^2} \cdot \frac{(x - a)^2}{(x - a)} = L \cdot 0 = 0 \). But,

\[
\frac{(x - a)^2}{(x - a)^2} \cdot \frac{(x - a)^2}{(x - a)} = \begin{cases} 1, & x \neq a, \\ \text{undefined}, & x = a. \end{cases}
\]

Thus by the “limits are a local property rule,” \( \lim_{x \to a} \frac{(x - a)^2}{(x - a)^2} \cdot \frac{(x - a)^2}{(x - a)} = 1 \). This is a contradiction.

\( \lim_{x \to a} (f/g)(x) \) does not exist (as a real number)

9 points; 3 points each.
To solve the next problem, we will need the following facts about the real numbers. We will not prove these facts in this class; to learn more, take Math 320 (in a few years), or google “the rational numbers are dense in the reals.”

Theorem 1: If \( x, y \in \mathbb{R} \) with \( x < y \), then there exists a rational number \( z \in \mathbb{Q} \) with \( x < z < y \). In other words, \((x, y) \cap \mathbb{Q} \) is not the empty set.

Theorem 2: If \( x, y \in \mathbb{R} \) with \( x < y \), then there exits an irrational number \( z \in \mathbb{R} \setminus \mathbb{Q} \) with \( x < z < y \). In other words, \((x, y) \cap (\mathbb{R} \setminus \mathbb{Q}) \) is not the empty set.

2. Considered the following rather strange function:

\[
f(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q}, \\
0, & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

For example, \( f(1/2) = 1 \), while \( f(\sqrt{2}) = 0 \).

a. What is the domain of \( f \)?
b. Prove that for every number \( a \in \mathbb{R} \) and every \( \delta > 0 \), there is a number \( x \in \mathbb{R} \) with \( 0 < |x - a| < \delta \) so that \( f(x) = 1 \).
c. Prove that for every number \( a \in \mathbb{R} \) and every \( \delta > 0 \), there is a number \( x \in \mathbb{R} \) with \( 0 < |x - a| < \delta \) so that \( f(x) = 0 \).
d. Prove that for every \( a \in \mathbb{R} \), \( \lim_{x \to a} f(x) \) does not exist (as a real number).

Solution.
a. \( D(f) = \mathbb{R} \)
b. Let \( p = a + \delta/3 \) and let \( q = a + \delta/2 \). Then \( a < p < q < a + \delta \). By Theorem 1 above, there exists \( x \in \mathbb{Q} \) with \( p < x < q \). This value of \( x \) satisfies \( 0 < |x - a| < \delta \), and \( f(x) = 1 \).
c. Let \( p = a + \delta/3 \) and let \( q = a + \delta/2 \). Then \( a < p < q < a + \delta \). By Theorem 2 above, there exists \( x \in \mathbb{R} \setminus \mathbb{Q} \) with \( p < x < q \). This value of \( x \) satisfies \( 0 < |x - a| < \delta \), and \( f(x) = 0 \).
d. Let \( a \in \mathbb{R} \) and let \( L \in \mathbb{R} \). Let \( \epsilon = 1/2 \). We will show that for each \( \delta > 0 \), there exists \( x \in \mathbb{R} \) satisfying \( 0 < |x - a| < \delta \) and \( |f(x) - L| \geq \epsilon \). In particular, this implies that \( \lim_{x \to a} f(x) \neq L \). Since this is true for every \( L \in \mathbb{R} \), this implies that \( \lim_{x \to a} f(x) \) does not exist (as a real number). Fix \( \delta > 0 \)

First, suppose \( L \leq 1/2 \). Then by part b), there exists \( x \in \mathbb{R} \) satisfying \( 0 < |x - a| < \delta \) with \( f(x) = 1 \) and thus \( |f(x) - L| \geq 1/2 = \epsilon \). Next, suppose \( L > 1/2 \). Then by part c), there exists \( x \in \mathbb{R} \) satisfying \( 0 < |x - a| < \delta \) with \( f(x) = 0 \) and thus \( |f(x) - L| \geq 1/2 = \epsilon \).

8 pts: a) 1 pt, b) 2 pts, c) 2 pts, d) 3 pts

3. Consider the function

\[
g(x) = \begin{cases} 
x^3, & \text{if } x \in \mathbb{Q}, \\
0, & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

a. Prove that \( \lim_{x \to 0} g(x) = 0 \).

b. Prove that for all \( a \in \mathbb{R} \setminus \{0\} \), \( \lim_{x \to a} g(x) \) does not exist (as a real number). (Hint: you can prove this directly, but it might be easier to do a proof by contradiction and use the limit rules discussed in class).

Solution.
a. This is a standard \( \epsilon-\delta \) proof. The key observation is that \( |g(x)| \leq |x|^3 \) for all \( x \in \mathbb{R} \). Now on to
the proof. First, note that since \( D(f) = \mathbb{R} \), the domain requirement is met. Next, let \( \varepsilon > 0 \). Select \( \delta = \varepsilon^{1/3} \); thus \( \delta > 0 \). Now, if \( 0 < |x - 0| < \delta \), then \( |g(x) - 0| \leq |x^3 - 0| = |x^3| < \delta^3 = \varepsilon \).

b. We will do a proof by contradiction. Let \( a \in \mathbb{R} \setminus \{0\} \) and suppose there exists a real number \( L \) so that \( \lim_{x \to a} g(x) = L \). Note as well (by the rule “\( \lim_{x \to a} x = a \)” and the product rule) that \( \lim_{x \to a} x^3 = a^3 \); since \( a \neq 0 \), \( a^3 \neq 0 \). By the quotient rule for limits (discussed in lecture), we have \( \lim_{x \to a} \frac{g(x)}{x^3} = \frac{L}{a^3} \).

Thus, if \( f(x) \) is the function from problem 2, then \( f(x) = g(x)/x^3 \) on the open interval \((a/2, 3a/2)\) (if \( a > 0 \)) or \((3a/2, a/2)\) (if \( a < 0 \)); in either case, this open interval contains \( a \). Thus by the “limits are a local property” rule, this implies \( \lim_{x \to a} f(x) = L/a^3 \). But this contradicts problem 2d, in which we proved that \( \lim_{x \to a} f(x) \) does not exist (as a real number).

Note that if \( g(a) = b \) and if \( f(b) = c \), then \( f \circ g(a) = c \). However, it need not be the case that if \( \lim_{x \to a} g(x) = b \) and \( \lim_{x \to b} f(x) = c \), the \( \lim_{x \to a} f \circ g(x) = c \). The next problem will ask you to find an example illustrating this point.

8 pts; 4 pts each.

4. Give an example of functions \( f \) and \( g \) so that \( \lim_{x \to 1} g(x) = 1 \), \( \lim_{x \to 1} f(x) = 1 \), but \( \lim_{x \to 1} f \circ g(x) \) is not equal to 1.

Solution. Let \( g(x) = 1 \). Then \( \lim_{x \to 1} g(x) = 1 \). Let

\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \setminus \{0\}; \\
0, & x \in \mathbb{R} \setminus \mathbb{Q} \\
\text{undefined}, & x = 0
\end{cases}
\]

Then by the “limits are a local property rule,” \( \lim_{x \to 1} f(x) = 1 \). On the other hand, \( f \circ g(x) = 10 \) for all \( x \in \mathbb{R} \), so by the limit rule for constant functions, \( \lim_{x \to 1} f \circ g(x) = 10 \).