Math 120 Homework 2 Solutions

Functions

1. Using the domain convention, state the domain and range of each of the following functions. You do not need to prove that your answer is correct
   a) $f(x) = x^2 - 1$
   b) $f(x) = \cos(1/x)$
   c) $f(x) = x^3 - \cos(1/x)$

Solution
   a. $D(f) = \mathbb{R}$, $R(f) = [-1, \infty]$.
   b. $D(f) = \mathbb{R}\{0\}$, $R(f) = [-1, 1]$.
   c. $D(f) = \mathbb{R}\{0\}$, $R(f) = \mathbb{R}$.

6 points; 1 point for domain and 1 point for range for each problem.

2. Let $f(x) = 1/x + 2$, $g(x) = \sin(x)$. What is the domain of $f \circ g$? You do not need to prove that your answer is correct.

Solution. Recall that $D(f \circ g) = \{x \in D(g): g(x) \in D(f)\}$. We have $D(g) = \mathbb{R}$ and $D(f) = \mathbb{R}\{0\}$. Note that $g(x) = 0$ if and only if $x$ is of the form $\pi n$ for some $n \in \mathbb{Z}$, i.e. $g(x) = 0$ if and only if $x \in \{2\pi n: n \in \mathbb{Z}\} = \ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots$.

Thus $D(f \circ g) = \mathbb{R}\{\pi n: \mathbb{Z}\} = \{x \in \mathbb{R}: x$ is not of the form $\pi n$ for some $n \in \mathbb{Z}\}$.

4 points; answers in degrees are also acceptable.

Universal Quantifiers

3. Write down the negation of each of the following expressions. You do not need to determine whether the expressions are true or false
   a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t. $x^2 > y$.
   b) $\forall x \in \mathbb{R}$ with $x > 0$, $\exists y \in \mathbb{R}$ s.t. $x > y^2$.
   c) $\forall x \in \mathbb{R}$ with $x > 0$, $\exists y \in \mathbb{R}$ s.t. $\forall z \in \mathbb{R}$, we have $x > y^2 - z$.
   d) $\forall \epsilon \in \mathbb{R}$ with $\epsilon > 0$, $\exists \delta \in \mathbb{R}$ with $\delta > 0$ s.t. $\forall x \in \mathbb{R}$ with $|x - 3| < \delta$, we have $|x^2 - 8| < \epsilon$.

Solution
   a. $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $x^2 \leq y$.
   b. $\exists x \in \mathbb{R}$ with $x > 0$s.t. $\forall y \in \mathbb{R}$, $x \leq y^2$.
   c. $\exists x \in \mathbb{R}$ with $x > 0$s.t. $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ s.t. $x \leq y^2 - z$.
   d. $\exists \epsilon > 0$ s.t. $\forall \delta \in \mathbb{R}$ with $\delta > 0$, $\exists x \in \mathbb{R}$ with $|x - 3| < \delta$ s.t. $|x^2 - 8| \geq \epsilon$.

8 points; 2 points each

Comment. The negations have to make sense as sentences! A common error was to leave phrases like “such that” at their place in the original sentence.
4. For each of the expressions from Question 3, determine whether the expression is true, or its negation is true, and (briefly) explain why.

Solution
a. The original expression is true: let \( y = x^2 - 1 \).
b. The original expression is true. Since \( x > 0 \), \( \sqrt{x} \) exists (as a real number). Let \( y = \sqrt{x}/2 \). Then \( y^2 = (\sqrt{x}/2)^2 = x/4 \), so \( x > y^2 \).
c. The negation is true. Let \( x = 1 \) (this satisfies \( x > 0 \)). Then for any \( y \in \mathbb{R} \), let \( z = y^2 - x \). Then \( x = y^2 - z \), so certainly \( x \leq y^2 - z \).
d. The negation is true. Let \( \epsilon = 0.1 \). Then for any \( \delta > 0 \), select \( x \in \mathbb{R} \) with \( |x - 3| \leq \delta \) and \( |x-3| < 0.1 \), so in particular, \( x > 2.9 \). Then \( x^2 > (2.9)^2 = 8.41 \), so \( |x^2-8| = x^2-8 > 8.41-8 = 0.41 \), which is larger than \( \epsilon \). 8 points; 2 points each

Comment. For d), a lot of people argued that this sentence means “the limit of \( x^2 \) when \( x \to 3 \) is 8.” This is technically not true, since for the limit, we say \( 0 < |x-3| < \delta \), while here it’s just \( |x-3| < \delta \).

Limits

5. Using the definition of a limit, prove that the following two statements are true.
a) \( \lim_{x \to 2} x^2 + x = 6 \).
b) \( \lim_{x \to 3} x^3 + x^2 + x = 3 \).

Hint: don’t forget to verify the domain requirement!

Solution
a. Let \( f(x) = x^2 + x \). Since \( D(f) = \mathbb{R} \), the domain requirement is met. Let \( \epsilon > 0 \). Select \( \delta = \min(\epsilon/6, 1) \). In particular, this means that if \( 0 < |x-2| < \delta \), then \( |x+2| \leq 5 \).

Next, if \( 0 < |x-2| < \delta \), we have

\[
|f(x) - 6| = |x^2 + x - 6| \\
= |(x^2 - 4) + (x - 2)| \\
\leq |x^2 - 4| + |x - 2| \\
= |x - 2| |x + 2| + |x - 2| \\
= (1 + |x + 2|) |x - 2| \\
\leq (1 + 5)|x - 2| \\
< 6\delta \\
\leq \epsilon.
\]

b. Let \( f(x) = x^3 + x^2 + x \). Since \( D(f) = \mathbb{R} \), the domain requirement is met. Let \( \epsilon > 0 \). Select \( \delta = \min(\epsilon/11, 1) \). In particular, this means that if \( 0 < |x-1| < \delta \), then \( |x+1| \leq 3 \) and \( |x^2+x+1| \leq 7 \).
Next, if \(0 < |x - 1| < \delta\), we have

\[
|f(x) - 3| = |x^3 + x^2 + x - 3| = |(x^3 - 1) + (x^2 - 1) + (x - 1)| \\
\leq |x^3 - 1| + |x^2 - 1| + |x - 1| \\
\leq |(x - 1)(x^2 + x + 1)| + |(x - 1)(x + 1)| + |(x - 1)| \\
= |x - 1| |x^2 + x + 1| + |x + 1| + |x - 1| \\
= (|x^2 + x + 1| + |x + 1| + 1) |x - 1| \\
\leq (7 + 3 + 1)|x - 1| \\
= 11 |x - 1| \\
< 11\delta \\
= 11\epsilon/11 \\
= \epsilon.
\]

6 points; 3 points each

*Comment.* It is important to state what \(\delta\) is at the beginning of the argument. I.e. begin with “Let \(\epsilon > 0\). Let \(\delta = \ldots\)” Some students stated what \(\delta\) is in the middle of the argument, or changed their \(\delta\) in the middle of computations. If you need to change your delta, then re-write the argument cleanly on a new piece of paper.

Also, the order of quantifiers is important. For example, when there is an \(\exists\) quantifier, it depends on what comes before, so \(\delta\) cannot depend on \(x\).

**6.** Using the definition of a limit, prove that the following two statements are false.

a) \(\lim_{x \to 10} 3x = 29\).

b) \(\lim_{x \to 1} \left(\frac{x^2}{x}\right) = 2\).

*Solution.*

a. Recall that in order to prove that a statement involving quantifiers is false, it is sufficient to prove that the negation is true. Thus we must show that \(\exists \epsilon > 0\) s.t. \(\forall \delta > 0 \exists x \in \mathbb{R}\) with \(|x - 10| < \delta\), and \(|3x - 10| \geq \epsilon\). We will select \(\epsilon = 0.5\). Then for any \(\delta > 0\), select \(x \in \mathbb{R}\) with \(|x - 10| < \delta\) and \(|x - 10| < 0.1\). Such an \(x\) must always exist, since if \(\delta < 0.1\) then we can select \(x = 10 - \delta/2\), while if \(\delta \geq 0.1\) then we can select \(x = 10.05\). In either case, we have \(|3x| > 3 \cdot 9.9 = 29.7\), so \(|3x - 29| > 0.7\), which is larger than \(\epsilon\).

b. Let \(\epsilon = 0.5\). Then for any \(\delta > 0\), select \(x \in \mathbb{R}\) with \(0 < |x - 1| < \delta\) and \(0 < |x - 1| < 0.1\), as argued above, such an \(x\) must always exist. Then since \(x \neq 0\), \(|x^2/x| = |x| < 1.1\), so \(|(x^2/x) - 2| \geq 0.9\), which is larger than \(\epsilon\).

*Note.* We didn’t discuss what happens if \(x\) is not in the domain of \(x^2/x\), so the grader was lenient when grading this.

6 points; 3 points each

**7.** Let \(f(x) = x\), \(g(x) = x\).

a) What is the domain of \((f/g)(x)\)?

b) Using the definition of a limit, prove that \(\lim_{x \to 0} (f/g)(x) = 1\).

*Solution.* \(D(f/g) = \mathbb{R}\setminus\{0\}\).

b. Let \(\epsilon > 0\). Let \(\delta = 1\). Then if \(0 < |x| < \delta\), we have \(|(f/g)(x) - 1| = |x/x - 1| = |1 - 1| = 0 < \epsilon\).

4 points (1 pt + 3 points).