Math 120 Homework 1 Solutions

Comments
Students often used a proof by contradiction, even when it was not necessary. i.e. students often had reasoning along the line of:

1) We want to show A.
2) For contradiction we assume not A
3) We prove A is true (without using not A)
4) Absurd
5) So A is true.
While this is a valid proof, Line (3) alone would have been enough.

Sets

1. Let $S = \{1, 2, 3, 4, 5\}$, $T = \{1, 3, 6\}$. What is
   a. $S \cap T$
   b. $S \cup T$
   c. $S \setminus T$
   d. $T \setminus S$

Solution.
a: $\{1, 3\}$
b: $\{1, 2, 3, 4, 5, 6\}$
c: $\{2, 4, 5\}$
d: $\{6\}$
(4 points; 1 point each).

Comments.
Some students were confused about $\cup$ vs $\cap$. If you are one of them, be sure to review the definitions.

2 a. Give examples of sets $S$ and $T$ where $S \setminus T = T \setminus S$.
b. Give examples of sets $S$ and $T$ where $S \setminus T \neq T \setminus S$.

Solution.
a: $S = T = \{1\}$. Then $S \setminus T = \emptyset$ and $T \setminus S = \emptyset$.
b: $S = \{1\}, T = \{2\}$. Then $S \setminus T = \{1\}$ and $T \setminus S = \{2\}$ so $S \setminus T \neq T \setminus S$.
(4 points; 2 points each)

3. Give an example of sets $S$ and $T$ so that $S$ and $T$ have infinitely many elements, but $S \cap T$ has finitely many elements.

Solution.
Let $S = \{n \in \mathbb{Z}: n \geq 0\}$, $T = \{n \in \mathbb{Z}: n \leq 0\}$. Since there are infinitely many non-negative integers, $S$ is infinite. Similarly, since there are infinitely many non-positive integers, $T$ is infinite. But $S \cap T = \{0\}$, which is finite. (2 points).

Comments.
Remember that an interval $[a, b] \subset \mathbb{R}$ with $a < b$ is an interval of finite length, but it is not a finite interval. This was a source of confusion for some students.
4. Recall that two sets are the same if they contain the same elements. Prove that if \( S \setminus T = \emptyset \) and \( T \setminus S = \emptyset \), then \( S = T \).

Solution.
We need to show that every element of \( S \) is also in \( T \), and vice-versa. First, let \( x \) be an element of \( S \). Since \( x \) is not an element of \( S \setminus T = \{ y \in S : y \text{ is not in } T \} \), we must have \( x \in T \). Thus every element of \( S \) is also in \( T \). An identical argument with \( S \) and \( T \) reversed shows that every element of \( T \) is also in \( S \). Thus \( S = T \). (3 points).

Upper bounds and least upper bounds

5. Let \( S \subset \mathbb{R} \). Prove that if \( x \in \mathbb{R} \) is an upper bound for \( S \), then \( x + 1 \) is also an upper bound for \( S \).

Solution.
We need to show that for every \( y \in S \), we have \( y \leq x + 1 \). Since \( x \) is an upper bound for \( S \), we know that for every \( y \in S \), we have \( y \leq x \). But since \( x < x + 1 \), we have \( y \leq x < x + 1 \), and thus \( y \leq x + 1 \) (indeed, the stronger statement \( y < x + 1 \) is true, but we don’t need this). Thus \( x + 1 \) is an upper bound for \( S \). (3 points)

6. Let \( S \subset \mathbb{R} \). Suppose that \( x \in \mathbb{R} \) is an upper bound for \( S \). Must it always be true that \( x - 1 \) is an upper bound for \( S \)? If so, then prove it. If not, then give an example of a set \( S \) and an upper bound \( x \) where \( x - 1 \) is not an upper bound for \( S \).

Solution.
No: Let \( S = \{ 1, 2, 3, 4, 5 \} \) and let \( x = 5 \). Then \( x \) is an upper bound for \( S \), but \( x - 1 = 4 \) is not an upper bound for \( S \) since \( 5 \in S \) and \( 5 > (x - 1) = 4 \). (2 points)

7. In this problem, we will use a proof by contradiction to prove that \( \sqrt{2} \) is not a rational number. Suppose that \( \sqrt{2} \) is a rational number, i.e. there exist integers \( m \) and \( n \) so that \( \frac{m}{n}^2 = 2 \). We can also suppose that \( m \) and \( n \) are not both even, since otherwise we could replace \( m \) and \( n \) by \( m/2 \) and \( n/2 \), and keep iterating this procedure until at least one of \( m \) and \( n \) is not even.

a. Prove that \( m \) must be even.

b. Prove that \( n \) must be even.

c. Explain why we have arrived at a contradiction to the assumption that \( \sqrt{2} \) is a rational number.

Solution
a. By assumption, \( 2 = (m/n)^2 \), where \( m \) and \( n \) are integers. Thus \( 2n^2 = m^2 \). Thus \( m^2 \) is even, so \( m \) is even.
b. Since \( m \) is even, it can be written as \( m = 2k \) for some integer \( k \). Thus \( 2n^2 = m^2 = (2k)^2 = 4k^2 \). Dividing by two, we have \( n^2 = 2k^2 \), so \( n^2 \) is even and thus \( n \) is even.
c. This is a contradiction because we assumed that there were integers \( m \) and \( n \) with \((m/n)^2 = 2 \). We noted that if such integers exist, then we can also find integers \( m \) and \( n \), not both of which are even, so that \( (m/n)^2 = 2 \). But we just showed that if \( (m/n)^2 = 2 \), then both \( m \) and \( n \) must be even, which is a contradiction.
(6 points—2 points each)

8. Consider the set of rational numbers that are positive and whose square is smaller than two, i.e. \( S = \{ z \in \mathbb{Q} : z > 0 \text{ and } z^2 < 2 \} \).

a. Give an example of a rational number that is an upper bound for \( S \).

b. Prove that if \( x \) is a rational number that is greater than zero, then \( (2x + 2)/(2 + x) \) is also a
rational number
c. Prove that if \( x \) is a rational number and if \( x \) is an upper bound for \( S \), then \( x^2 \geq 2 \). Hint: it might be useful to use the fact that if \( a > 0 \) and \( b > 0 \) are real rational numbers with \( a > b \), then \( a^2 > b^2 \) (this is true for real numbers as well, of course).
d. Prove that if \( x \) is a rational number with \( x > 0 \) and \( x^2 \geq 2 \), then \( x \) is an upper bound for \( S \).
e. Prove that if \( x \) is a rational number that is an upper bound for \( S \), then \( (2x + 2)/(2 + x) \) is also an upper bound for \( S \).
f. Prove that if \( x \) is a rational number that is an upper bound for \( S \), then \( (2x + 2)/(2 + x) < x \).

Solution
a. We will show that \( x = 2 \) is an upper bound for \( S \). This is because if \( z \in S \), then \( 2 - z = (2 - z)/(2 + z)/(2 + z) = (4 - z^2)/(2 + z) \) since \( z^2 < 2 \), and since \( 2 + z > 0 \), we have \( (4 - z^2)/(2 + z) < (4 - 2)/(2 + z) = 2/(2 + z) > 0 \), so \( 2 - z > 0 \) and thus \( z < 2 \).

Comments.
Very few students explained why their example was an upper bound (got only 1 point for writing a number with no explanation).

b. Assume \( x \) is rational, i.e. \( x = m/n \) with \( m, n \in \mathbb{Z} \) and \( n \neq 0 \). Note that \( m \) and \( n \) have the same sign since \( x > 0 \). In particular, we can take \( m \) and \( n \) to both be positive. Then
\[
\frac{2x + 2}{x} = \frac{2 + \frac{m}{2}}{2 + \frac{m}{n}} = \frac{2m + 2}{2n + m}.
\]
Since \( m \) and \( n \) are both positive, we have \( 2n + m \neq 0 \). Thus we have written \( \frac{2x + 2}{2 + x} \) as a quotient of two integers with non-zero denominator, so \( \frac{2x + 2}{2 + x} \) is rational.

Comments.
When students are writing \( (2 + 2x)/(2 + x) \) as a fraction, few mention that the denominator is not zero.

Some students plugged in specific values of \( x \) in \( (2 + 2x)/(2 + x) \) but do not prove the general case. This was also a problem in parts e and f.

c. We will prove this by contradiction. Suppose there existed a rational number \( x \) that is an upper bound for \( S \), but with \( x^2 < 2 \). First, since \( 1 \in S \), we must have \( x \geq 1 \), so in particular \( x > 0 \). Let \( y = (2x + 2)/(2 + x) \). By part b., \( y \) is a rational number. Since \( x > 0 \), \( y = (2x + 2)/(2 + x) > 0 \). Next, note that \( 2 - y^2 = 0 \), so \( y^2 < 2 \). This means that \( y \in S \).

Finally, note that \( y - x = \ldots > 0 \), so \( y > x \). This means that there exists an element of \( S \) (namely \( y \)) that is bigger than \( x \), which contradicts the assumption that \( x \) is an upper bound for \( S \).

Comments.
Almost everyone went with the following statement: If \( x \) is an upper bound and \( z \) is in \( S \), then \( x^2 \geq z^2 \) and \( z^2 < 2 \), so \( x^2 > 2 > z^2 \) with no explanation.

Equivalently, many people said that \( \sqrt{2} \) is the least upper bound of \( S \) with no justification.

d. We need to show that for every \( z \in S \), \( z < x \). But \( x - z = (x - z)/(x + z)/(x + z) = (x^2 - z^2)/(x + z) \). But since \( x^2 \geq 2 \) and \( z^2 < 2 \), and \( x > 0 \), \( z > 0 \), we have \( (x^2 - z^2)/(x + z) \geq (2 - z^2)/(x + z) > 0 \), so \( x > z \). Thus \( x \) is an upper bound for \( S \).

e. By part c, \( x^2 \geq 2 \). Let \( y = (2x + 2)/(2 + x) \). By part d, it suffices to show that \( y^2 > 2 \). But \( y^2 = (2x + 2)/(2 + x)^2 = 2^2 > 2 \).
f. By part c, \( x^2 \geq 2 \). Let \( y = (2x + 2)/(2 + x) \). We have \( x - y = \ldots > 0 \). Thus \( y < x \).