Math 120 Homework 8

- Due Monday Nov 21 at start of class.
- If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper. **Homework that is not stapled will lose 1 point.**
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

**Implicit differentiation**

1. (2 points) At which points on the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \)? Prove that your answer is correct.

**Proof.** The line \( y = x \) has slope 1. We need to find all points \((x, y)\) on the curve \( x^2 + 3y^2 = 1 \) where the curve has slope 1. If we regard \( y = y(x) \) as a function of \( x \), then we have

\[
0 = \frac{d}{dx}(x^2 + 3y^2) = 2x + 6y\frac{dy}{dx}
\]

re-arranging, we obtain

\[
\frac{dy}{dx} = -\frac{2x}{6y} = -\frac{x}{3y},
\]

whenever \( y \neq 0 \). Solving \(-x/3y = 1, x^2 + 3y^2 = 1\) we obtain the single equation \(12y^2 = 1\), or \(y = \pm 1/\sqrt{12}\). The corresponding points are \((\sqrt{3}/2, -1/\sqrt{12})\) and \((-\sqrt{3}/2, 1/\sqrt{12})\). However, this reasoning is only valid if \( y \neq 0 \). There are two points on the curve \( x^2 + 3y^2 = 1 \) where \( y = 0 \); the points are \((1, 0)\) and \((-1, 0)\). At both these points the tangent line to the curve \( x^2 + 3y^2 = 1 \) is vertical, so it is not parallel to the line \( y = x \).

Thus the two points where the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \) are \((\sqrt{3}/2, -1/\sqrt{12})\) and \((-\sqrt{3}/2, 1/\sqrt{12})\).

2. (3 points) Compute the slope of the tangent line of the curve \( y^5 + 2xy^3 + 3x^2y + 10x = 16 \) at the point \((1, 1)\).

**Proof.** If we regard \( y \) as a function of \( x \) and use implicit differentiation, then

\[
0 = \frac{d}{dx}(y^5 + 2xy^3 + 3x^2y + 10x) = 5y^4\frac{dy}{dx} + 2y^3 + 6xy^2\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} + 10.
\]

re-arranging, we get

\[
\frac{dy}{dx} = \frac{2y^3 + 6xy + 10}{5y^4 + 6xy^2 + 3x^2}
\]

whenever the denominator is non-zero. Plugging in \((x, y) = (1, 1)\), the denominator is \(5 + 6 + 3 = 14 \neq 0\), so we can compute

\[
\frac{dy}{dx} = \frac{2 + 6 + 10}{5 + 6 + 3} = \frac{9}{7}.
\]
Logarithmic differentiation

3. (6 points) Using the sum and difference rules for derivatives, the chain rule, and logarithmic differentiation, prove the product and quotient rules: if f, g are functions that are differentiable at c, then the function fg is differentiable at c and \((fg)'(c) = f'(c)g(c) + f(c)g'(c)\). Furthermore, if \(g(c) \neq 0\) then f/g is differentiable at c and \((f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}\).

Note: you are not allowed to use the product rule to solve the above problem!

**Proof.** We will first prove the product rule. Define \(F(x) = f(x) - f(c) + 1\), \(G(x) = g(x) - g(c) + 1\). Note that since f and g are differentiable (and thus continuous) at c, there exists \(\delta_1 > 0\) so that if \(|x - c| < \delta_1\) then \(|F(x) - F(c)| < 1\), and thus \(F(x) > 0\). Similarly, there exists \(\delta_2 > 0\) so that if \(|x - c| < \delta_2\) then \(G(x) > 0\). Let \(\delta = \min(\delta_1, \delta_2)\).

Observe as well that \(F(x)G(x) = f(x)g(x) + f(c)g(x) + f(x)g(c) + f(x)g(x)\). Thus by the sum rule for derivatives, for all x with \(|x - c| < \delta\) we have

\[
(FG)'(c) = (fg)'(c) - f(c)g'(c) - f'(c)g(c) + f'(c) + g'(c).
\]  

(1)

Now, since \(F(x) > 0\) and \(G(x) > 0\) for all x with \(|x - c| < \delta\), we log\((F(x))\) and log\((G(x))\) are well-defined for all x with \(|x - c| < \delta\). Thus

\[
\frac{F'(c)}{F(c)} + \frac{G'(c)}{G(c)} = \frac{d}{dx} \log(F(x)) + \log(G(x))|_{x=c} = \frac{d}{dx} \log(FG(x))|_{x=c} = \frac{(FG)'(c)}{FG(c)}
\]

Multiplying both sides by the non-zero quantity \(F(c)G(c)\), we obtain

\[
(FG)'(c) = F'(c)G(c) + F(c)G'(c).
\]

Since \(F'(c) = f'(c)\) and \(G'(c) = g'(c)\), this becomes

\[
(FG)'(c) = f'(c)(g(c) - g(c) + 1) + (f(c) - f(c) + 1)g'(c) = f'(c) + g'(c).
\]

Combining this with (1) above, we obtain

\[
f'(c) + g'(c) = (fg)'(c) - f(c)g'(c) - f'(c)g(c) + f'(c) + g'(c),
\]

so

\[
(fg)'(c) = f(c)g'(c) - f'(c)g(c)
\]

A similar computation gives the quotient rule.

Hyperbolic and inverse trigometric functions

4. (3 points) Compute the derivative of \(f(x) = \frac{\arccos x}{x^2 - 1}\). What is the domain of \(f(x)\)?

**Solution.** This is just an exercise in using the product and chain rule. The solution is

\[
\frac{\sqrt{1 - x^2} - 2x \arccos(x)}{(x^2 - 1)^2}
\]

(but you should show some work). This is well defined if the following conditions hold: \(1 - x^2 \geq 0\), i.e. \(x \in [-1, 1]\); \(x^2 - 1 \neq 0\); \(x \in D(\arccos x) = [-1, 1]\). Thus the domain of \(f'\) is \((-1, 1)\).
5. (3 points) Prove that $\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y)$ and $\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y)$.

Solution. We have

\[
\sinh(x + y) = \frac{1}{2}(e^{x+y} - e^{-x-y}) = \frac{1}{2}(e^x e^y - e^{-x} e^{-y}) = \frac{1}{4}((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y}))
\]

\[
= \sinh(x) \cosh(y) + \cosh(x) \sinh(y).
\]

A similar computation shows that $\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y)$.

L'Hopital's rule

6. (6 points) Let $f$ and $g$ be functions that are twice differentiable on the interval $[a, b]$, and suppose that $f''(x)$ and $g''$ are continuous on $[a, b]$. Let $c \in (a, b)$ and suppose $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Using Taylor's theorem with remainder (but not using L'Hopital's rule), prove that if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

Proof. By Taylor's theorem with remainder, we have

\[
f(x) = f(c) + f'(0)(x - c) + \frac{f''(x_1)}{2}(x - c)^2 = f'(c)(x - c) + \frac{f''(x_1)}{2}(x - c)^2,
\]

where $x_1$ is on the interval between $c$ and $x$. Similarly, we have

\[
g(x) = g(c) + g'(c)(x - c) + \frac{g''(x_2)}{2}(x - c)^2 = g'(c)(x - c) + \frac{g''(x_2)}{2}(x - c)^2,
\]

where $x_2$ is on the interval between $c$ and $x$. Thus if $x \neq c$,

\[
\frac{f(x)}{g(x)} = \frac{f'(c)(x - c) + \frac{f''(x_1)}{2}(x - c)^2}{g'(c)(x - c) + \frac{g''(x_2)}{2}(x - c)^2} = \frac{f'(c) + \frac{f''(x_1)}{2}(x - c)}{g'(c) + \frac{g''(x_2)}{2}(x - c)}.
\]

Since $f''(x)$ is continuous on $(a, b)$, there exists $\delta > 0$ so that if $|x - c| < \delta$, then $|f''(x) - f''(c)| < 1$, and thus if we define $M_1 = |f''(c)| + 1$ then $|f''(x)| < M_1$ for all $x$ with $|x - c| < \delta_1$. Thus for each $\epsilon > 0$, if we define $\delta = \min(\delta_1, \epsilon/M_1)$, then

\[
\left| \left( f'(c) + \frac{f''(x_1)}{2}(x - c) \right) - f'(c) \right| = |x - c| \left| \frac{f''(x_1)}{2} \right| \leq |x - c|M_1 \leq \epsilon.
\]

This means that

\[
\lim_{x \to c} f'(c) + \frac{f''(x_1)}{2}(x - c) = f'(c).
\]

An identical argument shows that

\[
\lim_{x \to c} g'(c) + \frac{g''(x_2)}{2}(x - c) = g'(c).
\]

Since by assumption $g'(c) \neq 0$, we can use the limit rule for quotients to conclude that

\[
\lim_{x \to c} \frac{f'(c) + \frac{f''(x_1)}{2}(x - c)}{g'(c) + \frac{g''(x_2)}{2}(x - c)} = \frac{f'(c)}{g'(c)}.
\]
Since $f$ and $g$ are twice differentiable, $f'(x)$ and $g'(x)$ are continuous, and since $g'(c) \neq 0$ we have

$$\frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Thus

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(c) + \frac{f''(x_1)}{2}(x - c)}{g'(c) + \frac{g''(x_2)}{2}(x - c)} = \frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$