Math 120 Homework 7

- Due Friday November 4 at start of class.

- If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper. **Homework that is not stapled will loose 1 point.**

- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

**An interesting function**

Recall that a polynomial is a function of the form \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \), where \( a_n, a_{n-1}, \ldots, a_0 \) are real numbers, and \( a_n \neq 0 \).

1. (4 points) Prove that if \( P(x) \) and \( Q(x) \) are polynomials, then their product \( P(x)Q(x) \) is a polynomial. Hint: you can prove this by induction on the degree of \( P \) (but you can also prove it some other way if you want.).

**Solution.** First, we will prove that the sum of two polynomials is a polynomial. If \( P(x) = a_n x^n + \ldots a_0 \) and \( Q(x) = b_m x^m + \ldots + b_m \), then for each \( j = 0, \ldots, \max(n,m) \),

\[
 c_j = \begin{cases} 
 a_j, & m < j \leq n, \\
 b_j, & n < j \leq m, \\
 a_j + b_j, & j \leq \min(m,n).
\end{cases}
\]

Let \( \ell \) be the largest index \( j \) for which \( c_j \) is non-zero. Then \( P(x) + Q(x) = c_\ell x^\ell + \ldots + c_0 \).

Next we will prove that the product of two polynomials is a polynomial; we will prove the result by induction on the degree of \( P \). First, if \( P \) has degree 0, then \( P(x) = a_0 \). If \( Q(x) = b_m x^m + \ldots + b_0 \), then \( P(x)Q(x) = a_0 b_m x^m + a_0 b_{m-1} x^{m-1} + \ldots + a_0 b_0 \), which is a polynomial. Now, suppose the result has been proved whenever \( P \) is a polynomial of degree at most \( n \). Let \( P \) be a polynomial of degree \( n+1 \), i.e. \( P(x) = a_{n+1} x^{n+1} + a_n x^n + \ldots + a_0 \). Then we can write \( P = a_{n+1} x^{n+1} + P_1(x) \), where \( P_1(x) = a_n x^n + \ldots + a_0 \) is a polynomial of degree at most \( n \) (note that \( P_1 \) might be of degree less than \( n \), since \( a_n \) might be 0). Then \( P(x)Q(x) = a_{n+1} x^{n+1} Q(x) + P_1(x) Q(x) \). Since \( P_1 \) has degree at most \( n \), we know that \( P_1(x) Q(x) \) is a polynomial. Similarly, \( a_{x+1} x^{n+1} Q(x) = a_{n+1} b_m x^{n+m+1} + \ldots + a_{c+1} b_0 x^{n+1} \) is a polynomial, and we showed above that the sum of two polynomials is a polynomial. Thus \( P(x)Q(x) \) is a polynomial, which completes the induction step.

2. (6 points) Let \( f(x) = e^{-1/x} \) if \( x > 0 \) (i.e. \( D(f) = (0, \infty) \)). Prove by induction that for every integer \( n \geq 1 \), there exists a polynomial \( P_n(x) \) so that \( f^{(n)}(x) = P_n(1/x) e^{-1/x} \).

**Solution.** We will actually prove the result for all \( n \geq 0 \). The base case \( n = 0 \) is trivial: let \( P_0(x) = 1 \). Now suppose we know that there is a polynomial \( P_n \) so that \( f^{(n)}(x) = P_n(1/x) e^{-1/x} \);
we will prove the result for \( n + 1 \). We have
\[
f^{(n+1)}(x) = (f^{(n)})'(x) = (P_n(1/x)e^{-1/x})' = P'_n(1/x)(-1/x^2)e^{-1/x} + P_n(1/x)e^{-1/x}(1/x^2) = (P'_n(1/x)(-1/x^2) + P_n(1/x)(1/x^2))e^{-1/x}.
\]
Thus if we define \( P_{n+1}(y) = P'_n(y)(-y^2) + P_n(y)y^2 \), then \( f^{(n+1)}(x) = P_{n+1}(1/x)e^{-1/x} \). By problem 1, \( P'_n(y)(-y^2) \) and \( P_n(y)y^2 \) are polynomials, and the sum of two polynomials is also a polynomial, so \( P_{n+1}(y) \) is a polynomial. This completes the induction step.

3. (6 points) Let \( f(x) \) be the function from question 2. Prove that for every integer \( n \geq 1 \), \( \lim_{x \to 0^+} f^{(n)}(x) = 0 \). Hint: HW3 #2 might be helpful.

Solution. First, recall from lecture that if \( P(x) \) is a polynomial, then \( \lim_{x \to \infty} P(x)e^{-x} = 0 \). Let \( g(x) = 1/x \). By HW3 #2, we have that for each \( n \geq 1 \),
\[
0 = \lim_{x \to \infty} P_n(x)e^{-x} = \lim_{x \to 0^+} P_n(1/x)e^{-1/x} = \lim_{x \to 0^+} f^{(n)}(x).
\]

4. (6 points) Let
\[
g(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0 \end{cases}
\]
Prove that \( g \) is infinitely differentiable on \( \mathbb{R} \), i.e. for every integer \( n \geq 0 \) and every \( x \in \mathbb{R} \), \( g^{(n)}(x) \) exists.

Solution. Let \( t(x) = 0 \). If \( x < 0 \) then by the limits are a local property rule, we have \( g^{(n)}(x) = t^{(n)}(x) = 0 \), so in particular, \( g(x) \) is infinitely differentiable at every point \( x < 0 \). Similarly, if \( x > 0 \) the by the limits are a local property rule, we have \( g^{(n)}(x) = f^{(n)}(x) = P_n(1/x)e^{-1/x} \) so again, \( g(x) \) is infinitely differentiable at every point \( x > 0 \). It remains to prove that \( g(x) \) is infinitely differentiable at \( x = 0 \). We will prove by induction on \( n \) that \( g^{(n)}(0) = 0 \) for all \( n \). We will begin with \( n = 0 \). Since \( h(x) \) is defined for all \( x \in \mathbb{R} \), \( g(x) \) is 0-times differentiable at 0, and \( g^{(0)}(0) = g(0) = 0 \). Now suppose we have shown that \( g^{(n)}(0) = 0 \). In order to prove that \( g^{(n+1)}(0) = 0 \), it suffices to prove that
\[
\lim_{h \to 0^-} \frac{g^{(n)}(h) - g^{(n)}(0)}{h} = \lim_{x \to 0^+} \frac{g^{(n)}(h) - g^{(n)}(0)}{h} = 0.
\]
Since \( g^{(n)}(0) = 0 \) and \( g^{(n)}(h) = 0 \) for all \( h < 0 \), we have \( \lim_{h \to 0^-} \frac{g^{(n)}(h) - g^{(n)}(0)}{h} = 0 \). Since \( g^{(n)}(h) = f^{(n)}(h) \) for all \( h > 0 \), by problem 3 we have \( \lim_{x \to 0^+} \frac{g^{(n)}(h) - g^{(n)}(0)}{h} = 0 \). This completes the proof.

5. (3 points) Recall the definition of the degree \( n \) Taylor expansion of a function around a point \( c \in \mathbb{R} \) from HW6. For each integer \( n \geq 1 \), compute the degree \( n \) Taylor expansion of \( g(x) \) around the point \( c = 0 \).

Let \( P_n(x) \) be the degree \( n \) Taylor expansion of \( g(x) \) around 0. We have
\[
P_n(x) = g(0) + \sum_{j=1}^{n} \frac{g^{(j)}(0)}{j!}x^j = 0 + \sum_{j=1}^{n} 0 \cdot x^j = 0.
\]
Remark: This is strange: The function \( g(x) \) from problem 4 and the function \( h(x) = 0 \) are both smooth, and at the point \( c = 0 \), all of their derivatives agree. But the two functions are not equal to each other...
A question asked in class

The following problems are optional, bonus problems. Thus, it is possible to score as high as 37/25 = 148% on this homework.

Let $f$ be a function with $D(f) = \mathbb{R}$ and suppose that $f$ is differentiable at every point $x \in \mathbb{R}$. Suppose there exist points $a, b \in \mathbb{R}$ with $f'(a) > 0$, $f'(b) < 0$. We will prove that $f$ is not one-to-one. For simplicity, we will assume $a < b$ (the case $a > b$ is similar; just replace $f(x)$ with $f(-x)$). The following questions will be about this function $f$.

6. (2 points) For $t \in (a, b]$, define $S_t = \{f(x) : a \leq x < t\}$. Prove that if $f$ (described above) is one-to-one, then for each $t \in [a, b]$, $f(t) \notin S_t$.

Proof. Since We will prove the result by contradiction. Suppose $f(t) \in S_t$. We must show that $f$ fails to be one-to-one. But if $f(t) \in S(t)$, then there exists some $x \in [a, t)$ so that $f(x) = f(t)$. Since $x \notin [a, t)$, this means $x$ and $t$ are distinct, but $f(x) = f(t)$, which is a contradiction.

7. (5 points) Prove that if $f$ (described above) is one-to-one, then for each $t \in (a, b]$, $S_t = [f(a), f(t)]$. Hint: this is not so easy.

Solution. First, suppose $f(t) > f(a)$. Since $f$ is differentiable, it is continuous, so by the intermediate value theorem applied to $f$ on the interval $[a, t]$, we have that for every value of $y$ between $f(a)$ and $f(t)$, there is a value of $x$ between $a$ and $t$ so that $f(x) = y$. In particular, this means that $[f(a), f(t)) \subset S_t$. Now suppose $S_t \neq [f(a), f(t))$. This means there exists a point $y \in S_t \setminus [f(a), f(t))$. Let $x_0 \in [a, t)$ with $f(x_0) = y$. If $y < f(a)$, then by the intermediate value theorem, there exists a point $x \in (x_0, t)$ with $f(x) = f(a)$, and this contradicts the fact that $f$ is one-to-one. If $y = f(t)$, then this contradicts the fact that $f$ is one-to-one. If $y > f(t)$, then by the intermediate value theorem there exists a point $x \in (a, x_0)$ with $f(x) = f(t)$, and this again contradicts the fact that $f$ is one-to-one. Thus if $f(t) > f(a)$, then $S_t = [f(a), f(t))$.

Next, suppose $f(t) < f(a)$. Since $f'(a) > 0$, there exists a number $\delta > 0$ so that for all $x \in [a, a + \delta)$, $f(x) > f(a)$. In particular, $f(a + \delta/2) > f(a)$. Thus by the intermediate value theorem, there exists a number $x \in (a + \delta/2, t)$ with $f(x) = f(a)$, and this again contradicts the fact that $f$ is one-to-one.

We conclude that $S_t = [f(a), f(t))$.

8 (5 points) Prove that $f$ (described above) cannot be one-to-one.

Solution. Since $f'(b) < 0$, there exists a number $\delta > 0$ so that for all $x, y \in (b - \delta, b)$, if $x < y$ then $f(x) > f(y)$. But $S_x = [f(a), f(x))$ and $S_y = [f(a), f(y))$. If $f(x) > f(y)$, then by problem 7, $S_y \subset S_x$, so for every number $z$ with $x < z < y$, there is a number $w \in [a, x)$ with $f(z) = f(w)$, and thus $f$ cannot be one-to-one.