Math 120 Homework 6 Solutions

• Due Friday October 21 at start of class.

• If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper. **Homework that is not stapled will loose 1 point.**

• Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

**Landau’s Big O notation**

1. (2 points) Prove that \( x^3 + x^2 + 10 = O(1) \) as \( x \to 0 \).

   **Proof.** Let \( M = 12, \delta = 1 \). Observe that if \( |x| \leq 1 \), then \( |x^3| \leq 1 \), and similarly if \( |x| \leq 1 \) then \( |x|^2 \leq 1 \). Thus if \( |x-0| < \delta = 1 \), then \( |x^3 + 2x^2 + 10| \leq |x^3| + |x^2| + 10 \leq 1 + 1 + 10 \leq 12 \leq M|1| \).

b. (2 points) Prove that \( x^2 + x = O(1) \) as \( x \to 0 \).

   **Proof.** Let \( M = 2, \delta = 1 \). Observe that if \( |x| \leq 1 \), then \( |x^2| \leq |x| \). Thus if \( |x-0| < \delta = 1 \), then \( |x^2 + x| \leq |x^2| + |x| \leq |x| + |x| = 2|x| \leq M|x| \).

c. (2 points) Prove that \( x^2 + x = O(1) \) as \( x \to 0 \).

   **Proof.** Let \( M = 2, \delta = 1 \). Observe that if \( |x| \leq 1 \), then \( |x^2| \leq 1 \) and \( |x| \leq 1 \). Thus \( |x^2 + x| \leq |x^2| + |x| \leq 1 + 1 \leq 2 \leq M|1| \).

d. (2 points) Prove that the statement “\( 1 = O(2) \) as \( x \to 0 \)” is false.

   **Proof.** Suppose that \( 1 = O(2) \) as \( x \to 0 \) (we will arrive at a contradiction). This means that there exists a number \( M \) and a number \( \delta > 0 \) so that \( 1 \leq M|x^2| \) whenever \( |x-0| < \delta \). If \( M \leq 0 \), then this is clearly false; just take \( x = \delta/2 \). Now suppose \( M > 0 \). Let \( x = \min(\delta, 1/\sqrt{M}) \). Then \( M|x^2| < 1 \), which contradicts the assumption that \( 1 \leq M|x^2| \).

2. (2 points) Prove that \( \cos(x) = O(1) \) as \( x \to \infty \)

   **Proof.** Let \( R = 0, M = 1 \). Then for all \( x \in \mathbb{R} \) (and in particular, for all \( x \geq 0 \)), we have \( |\cos(x)| \leq 1 \leq M|1| \).

b. (2 points) Prove that \( x^2 + \sin(3x) + 1 = O(x^2) \) as \( x \to \infty \).

   **Proof.** Let \( R = 1, M = 3 \). Then for all \( x \geq 1 \), we have \( |x^2| \geq 1 \) and \( |\sin(3x)| \leq 1 \), so \( |x^2 + \sin(3x) + 1| \leq |x^2| + |\sin(3x)| + |1| \leq |x^2| + |x^2| + |x^2| \leq M|x^2| \).

c. (2 points) Prove that \( x^2 + \sin(3x) + 1 = O(x^3) \) as \( x \to \infty \).

   **Proof.** Let \( R = 1, M = 3 \). Then for all \( x \geq 1 \), \( |x^2| \leq |x^3| \). We already showed that \( |x^2 + \sin(3x) + 1| \leq 3|x^2| \) if \( x > 1 \), so \( |x^2 + \sin(3x) + 1| \leq 3|x^2| \leq 3|x^3| \) if \( x > 1 \).

3. (4 points) Prove that if \( f(x) = O(g(x)) \) as \( x \to c \) and \( g(x) = O(h(x)) \) as \( x \to c \), then \( f(x) = O(h(x)) \) as \( x \to c \).
Proof. Since \( f(x) = O(g(x)) \) as \( x \to c \) there exists \( M_1 \) and \( \delta_1 > 0 \) so that \( |f(x)| \leq M_1|g(x)| \) whenever \( |x - c| < \delta_1 \). Similarly, there exists \( M_2 \) and \( \delta_2 \) so that \( |g(x)| \leq M_2|h(x)| \) whenever \( |x - c| < \delta_2 \). Let \( M = M_1M_2 \) and let \( \delta = \min(\delta_1, \delta_2) \). Then whenever \( |x - c| < \delta \), we have
\[
|f(x)| \leq M_1|g(x)| \leq M_1M_2|h(x)|.
\]

4. (4 points) Prove that if \( f \) and \( g \) are functions whose domain is \( \mathbb{R} \), and \( f = O(g) \), then the statement \( \lim_{x \to \infty} g(x)/f(x) = 0 \) is false.

Proof. Since \( f(x) = O(g(x)) \), there exists \( R \) and \( M \) so that \( |f(x)| \leq M|g(x)| \), i.e. \( |g(x)/f(x)| \geq 1/M \) for all \( x \geq R \). Now, suppose that \( \lim_{x \to \infty} g(x)/f(x) = 0 \). This would mean that for all \( \delta > 0 \), there exists \( R_1 \) so that if \( x > R_1 \), then \( |g(x)/f(x)| \leq \delta \). We will show that this cannot happen. Indeed, let \( \delta = 1/M \). Then for all \( R_1 > 0 \), there exists \( x > R_1 \) so that \( |g(x)/f(x)| > \delta \); just take \( x = \min(M, M_1) \).

Taylor polynomials

5 a. (5 points) Prove by induction that if \( k \) and \( n \) are integers and \( 1 \leq k \leq n \), and if \( f(x) = x^n \), then \( f^{(k)}(x) = n(n-1)(n-2) \cdots (n-k+1)x^{n-k} \).

Proof. For each value of \( n \), we will prove the result by induction on \( k \). First, if \( k = 1 \) then \( f(x) = (x^n)' = nx^{n-1} \); this establishes the base case \( k = 1 \). Now suppose (for a fixed value of \( n \)) that we have proved that \( f^{(k)}(x) = n(n-1)(n-2) \cdots (n-k+1)x^{n-k} \). If \( k = n \) then there is nothing more to prove. If \( k < n \), then
\[
f^{(k+1)}(x) = (f^{(k)})'(x) = (n(n-1)(n-2) \cdots (n-k+1)x^{n-k})'
\]
\[
= n(n-1)(n-2) \cdots (n-k+1)(n-k)x^{n-k-1}
\]
\[
= n(n-1)(n-2) \cdots (n-k+1)((n-k)x^{n-k-1})
\]
\[
= n(n-1)(n-2) \cdots (n-k+1)((n-k+1)x^{n-(k+1)}).
\]
This establishes the induction step, and completes the proof.

b. (3 points) Prove that if \( k > n \) and if \( f(x) = x^n \), then \( f^{(k)}(x) = 0 \). Hint: part a might be helpful.

Proof. First, by part a, we have \( f^{(n)}(x) = n! \), and thus \( f^{(n+1)}(x) = 0 \). By definition, if \( k > n \) then
\[
f^{(k)}(x) = \left( \frac{d^k}{dx^k} f \right)(x) = \left( \frac{d^{k-n-1}}{dx^{k-n-1}} \left( \frac{d^{n+1}}{dx^{n+1}} f \right) \right)(x) = \left( \frac{d^{k-n-1}}{dx^{k-n-1}}(0) \right)(x) = 0,
\]
so \( f^{(k)}(x) = 0 \).

For the next few problems, we will use the following definition. Let \( f \) be a function that is \( n \) times differentiable on \( [a,b] \). Let \( c \in [a,b] \). We define the “degree \( n \) Taylor expansion of \( f \) around \( c \)” to be the function
\[
h(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.
\]

6. (3 points) Let \( f(x) \) and \( g(x) \) be functions that are \( n \)-times differentiable on \( [a,b] \). Let \( c \in [a,b] \). Let \( f_n \) and \( g_n \) be the \( n \)-th order Taylor expansions of \( f \) and \( g \) around the point \( c \), respectively. Prove that \( (f_n + g_n)(x) \) is the \( n \)-th order Taylor expansion of \( f + g \) around the point \( c \).
Proof. This is just a repeated application of the derivative rule “ \((f + g)'(x) = f'(x) + g'(x)\),” and thus \((f + g)^{(k)}(x) = f^{(k)}(x) + g^{(k)}(x)\).

Let \(h = f + g\), and let \(h_n\) be the degree \(n\) Taylor expansion of \(f\) around \(c\). Then

\[
f_n + g_n = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k + g(c) + \sum_{k=1}^{n} \frac{g^{(k)}(c)}{k!} (x - c)^k
\]

\[
= f(c) + g(c) + \sum_{k=1}^{n} \left[ \frac{f^{(k)}(c)}{k!} (x - c)^k + \frac{g^{(k)}(c)}{k!} (x - c)^k \right]
\]

\[
= (f + g)(c) + \sum_{k=1}^{n} \frac{(f + g)^{(k)}(c)}{k!} (x - c)^k
\]

\[
= h_n(x).
\]

5. (4 points) Compute the degree 3 Taylor expansion of \(f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 1\) around the point \(x = 0\). You don’t need to prove that your answer is correct, but do show your work.

Proof. The Taylor expansion is

\[1 + 5x + 4x^2 + 3x^3\]

(but when you do it, you need to show your work.)

6. (4 points) Compute the degree 3 Taylor expansion of \(f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 1\) around the point \(x = 2\). You don’t need to prove that your answer is correct, but do show your work.

Proof. The Taylor expansion is

\[115 + 201(x - 2) + 150(x - 2)^2 + 59(x - 2)^3\]

(but when you do it, you need to show your work.)