Math 120 Homework 4

- Due Friday October 14 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper. **Homework that is not stapled will loose 1 point.**
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Intermediate value theorem

1 a. (5 points) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be a polynomial. Suppose that $a_n > 0$ and $a_0 < 0$. Prove that there exists a point $x \in \mathbb{R}$ with $f(x) = 0$.

b. (2 points) Give an example of a non-constant polynomial f (i.e. the polynomial must have degree at least one) so that $f(x) \neq 0$ for all $x \in \mathbb{R}$. You do not have to prove that your example is correct.

Derivatives

2 a. (2 points). Suppose that f is continuous at c, and g is not continuous at c. Prove that $f + g$ is not continuous at c. (Hint: there is a 1 line proof)
b (2 points). Suppose that f is differentiable at c, and g is not differentiable at c. Prove that $f + g$ is not differentiable at c. (Hint: there is a 1 line proof)

3. (4 points) We will use the following fact: if $x \neq 1$, then

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}. \tag{1}$$

Using Equation 1, write down a formula for $1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$ which is valid for all values $x \neq 1$.

Using the chain rule

In lecture, we will prove that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$, and if $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. You may use these facts freely.
4. (5 points) Let $f(x) = \tan(x^2)$. Using the derivative rules discussed in class (chain rule, product rule, quotient rule, etc.), compute $f'(x)$. Be sure to show your work.

5. a (3 points) Let

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that f is differentiable at every point $x \in \mathbb{R}$ (and thus $D(f') = \mathbb{R}$).

b. (4 points) Prove that f' is continuous at every point $x \neq 0$, and is discontinuous at $x = 0$.

c. (3 points) Give an example of a function that is differentiable at every point $x \in \mathbb{R}$, but whose derivative f' is discontinuous on the set $\{1, 2, 3\} \subset \mathbb{R}$. (hint: parts a and b, as well as 1a might be useful)