Math 120 Homework 3

- Due Friday September 30 at start of class.
- If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper. Homework that is not stapled will lose 1 point.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Limits

1. (3 points) Give an example of a function \( f(x) \) for which \( \lim_{x \to 0^+} f(x) = 1 \) exists, but for which the statement \( \lim_{x \to 0^-} f(x) = L \) is false for every real number \( L \). Prove that your answer is correct.

2. This problem shows a relationship between one-sided limits and limits at infinity. Let \( g(x) = \frac{1}{x} \).
   Let \( f(x) \) be a function and suppose that \((0, \infty) \subset D(f)\).
   
   a. (4 points) Suppose that \( \lim_{x \to 0^+} f(x) = L \). Prove that \( \lim_{x \to \infty} f \circ g(x) = L \) (don’t forget to establish both requirements for the limit to exist).
   
   b. (4 points) Suppose that \( \lim_{x \to \infty} f \circ g(x) = L \). Prove that \( \lim_{x \to 0^+} f(x) = L \).

In class, we established limit rules for the sum, difference, product, and quotient of limits. These results also hold for one-sided limits. In the case of products, we have:

**Theorem 1 (Product rule for one-sided limits)** Let \( f \) and \( g \) be functions, \( a \in \mathbb{R} \), and suppose \( \lim_{x \to a^+} f(x) = L_1 \), \( \lim_{x \to a^+} g(x) = L_2 \). Then \( \lim_{x \to a^+} (fg)(x) = L_1 L_2 \).

3. (5 points) Suppose that
   \[
   \lim_{x \to \infty} \frac{\sqrt{x + 1}}{\sqrt{2x + 1}} = L
   \]
   for some real number \( L \) (i.e. you may assume that the limit exists and is a real number). Using Problem 2 and the product rule for one-sided limits, prove that
   \[
   \lim_{x \to \infty} \frac{\sqrt{x + 1}}{\sqrt{2x + 1}} = \frac{1}{\sqrt{2}}.
   \]

Continuity

4. (6 points) Let \( f \) and \( g \) be functions, and suppose \( f \) and \( g \) are continuous (in particular, this means \( D(f) = \mathbb{R} \), \( D(g) = \mathbb{R} \)). Suppose that for every rational number \( x \in \mathbb{Q} \), we have \( f(x) = g(x) \). Prove that \( f(x) = g(x) \) for every number \( x \in \mathbb{R} \).

5. Define
   \[
   f(x) = \begin{cases} 
   1/p^2, & \text{if } x = q/p \text{ in lowest form,} \\
   0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}.
   \end{cases}
   \]
a. (4 points) Prove that if $a \in \mathbb{Q}$, then $f$ is not continuous at $a$.

b. (4 points) prove that if $a \in \mathbb{R} \setminus \mathbb{Q}$, then $f$ is continuous at $a$. 