Math 120 Homework 2 Solutions

Limits

1. In lecture, we discussed the limit rule: If \( \lim_{x \to a} f(x) \) is a real number, and \( \lim_{x \to a} g(x) \) is a non-zero real number, then
   \[
   \lim_{x \to a} (f/g)(x) = \left( \lim_{x \to a} f(x) \right) / \left( \lim_{x \to a} g(x) \right).
   \]
   However, if \( \lim_{x \to a} g(x) = 0 \), then all bets are off. Give an example of functions \( f, g \) with \( \lim_{x \to a} f(x) = 0, \lim_{x \to a} g(x) = 0 \), and:
   
   a (3 points) \( \lim_{x \to a} f/g(x) = 1 \).

   proof Let \( f(x) = x - a, g(x) = x - a \). By the difference rule for limits, \( \lim_{x \to a} f(x) = \lim_{x \to a} x - \lim_{x \to a} a \). By the limit rule “\( \lim_{x \to a} x = a \)” and by the limit rule “\( \lim_{x \to a} K = K, \)” we have \( \lim_{x \to a} f(x) = a - a = 0 \). Similarly, \( \lim_{x \to a} g(x) = 0 \). Thus \( f \) and \( g \) satisfy the hypotheses of the problem.

   On the other hand, \( (f/g)(x) = 1 \), if \( x \neq 0 \). Let \( h(x) = 1 \). Then \( (f/g)(x) = h(x) \) for \( x \in \mathbb{R}\{a\} \).

   Thus by the “limits are a local property rule,” and by the limit rule “\( \lim_{x \to a} K = K, \)” we have \( \lim_{x \to a} (f/g)(x) = \lim_{x \to a} 1 = 1 \).

   b (3 points) \( \lim_{x \to a} f/g(x) = 0 \).

   proof Let \( f(x) = (x - a)^2, g(x) = x - a \). We already proved that \( \lim_{x \to a} g(x) = 0 \). By the product rule, \( \lim_{x \to a} f(x) = \left( \lim_{x \to a} (x - a) \right) \left( \lim_{x \to a} (x - a) \right) = (0)(0) = 0 \), so \( f \) and \( g \) satisfy the hypotheses of the theorem. Now, observe that \( (f/g)(x) = (x - a) \) if \( x \neq a \). Thus by the “limits are a local property rule,” \( \lim_{x \to a} (f/g)(x) = \lim_{x \to a} (x - a) = 0 \).

   c (3 points) For every real number \( L \), the statement “\( \lim_{x \to a} (f/g)(x) = L \)” is false.

   proof Let \( f(x) = (x - a), g(x) = (x - a)^2 \). We already proved in part b that \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \) (the roles of \( f \) and \( g \) were reversed, but the proof is the same). Suppose that \( \lim_{x \to a} \frac{(x-a)}{(x-a)^2} = L \) for some real number \( L \). Then by the product rule for limits and part b above, \( \lim_{x \to a} \frac{(x-a)^2}{(x-a)} \cdot \frac{(x-a)^2}{(x-a)} = L \cdot 0 = 0 \). But,
   \[
   \frac{(x-a)}{(x-a)^2} \cdot \frac{(x-a)^2}{(x-a)} = \begin{cases} 1, & x \neq a, \\ \text{undefined,} & x = a \end{cases}.
   \]

   Thus by the “limits are a local property rule,” \( \lim_{x \to a} \frac{(x-a)^2}{(x-a)^2} = 1 \). This is a contradiction. Thus the statement “\( \lim_{x \to a} \frac{(x-a)^2}{(x-a)^2} = L \)” is false for every real number \( L \).

2. In this problem we will study the function
   \[
   f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}
   \]
Thus \( f(x) = 1 \) if \( x \) is rational, and \( f(x) = 0 \) if \( x \) is irrational.

### a. (2 points) Prove that if \( q/p \in \mathbb{Q} \), \( q \neq 0 \), and \( w \in \mathbb{R} \setminus \mathbb{Q} \), then \( w \cdot q/p \in \mathbb{R} \setminus \mathbb{Q} \).

**Proof.** Suppose that \( w \cdot q/p \in \mathbb{Q} \). This means it can be written in the form \( w \cdot q/p = a/b \), where \( a \) and \( b \) are integers and \( b \neq 0 \). But this implies that \( w = \frac{aq}{bp} \). Since \( q, p, a, \) and \( b \) are integers, \( ap \) and \( bq \) are integers. Finally, since neither \( b \) nor \( q \) is 0, \( bq \) is non-zero. Thus we have written the irrational number \( w \) as a quotient of two integers, which is a contradiction. We conclude that \( w \cdot q/p \in \mathbb{R} \setminus \mathbb{Q} \).

### b. (2 points) Prove that if \( s \) and \( t \) are real numbers with \( s < t \), then there exists a number \( w \in \mathbb{R} \setminus \mathbb{Q} \) with \( s < w < t \). (Hint: HW 1 #6 might be helpful, though you can also prove this fact directly.)

**Proof.** Since \( s < t \) and \( 1/\sqrt{2} > 0 \), we have \( s/\sqrt{2} < t/\sqrt{2} \). By HW 1 # 6c, we can find a rational \( q/p \) so that \( s/\sqrt{2} < q/p < t/\sqrt{2} \). Since \( \sqrt{2} > 0 \), this implies \( s < \sqrt{2} \cdot q/p < t \). By part a, we know that \( \sqrt{2} \cdot q/p \notin \mathbb{Q} \).

### c. (5 points) Prove that for all \( a \in \mathbb{R} \) and all \( L \in \mathbb{R} \), the statement “\( \lim_{x \to a} f(x) = L \)” is false.

**Proof.** Let \( a \in \mathbb{R} \) and suppose there exists some \( L \in \mathbb{R} \) so that \( \lim_{x \to a} f(x) = L \). Let \( \epsilon = 1/3 \). Then for each \( \delta > 0 \), apply part b to conclude that there exists a real number \( x \in \mathbb{R} \setminus \mathbb{Q} \) with \( a < x < a + \delta \), so in particular, \( 0 < |x - a| < \delta \).

Since \( x \in \mathbb{R} \setminus \mathbb{Q} \), we have \( f(x) = 0 \), so if \( |f(x) - L| < \epsilon \), then \( |0 - L| < 1/3 \), i.e. \(-1/3 < L < 1/3 \). Next, apply problem 6c from HW 1 to find a number \( x \in \mathbb{Q} \) with \( a < x < a + \delta \). Since \( x \in \mathbb{Q} \), \( f(x) = 1 \), so if \( |f(x) - L| < \epsilon \) then \( |1 - L| < 1/3 \), i.e. \( 2/3 < L < 4/3 \). But this means \( L < 1/3 \) and \( L > 2/3 \), which is a contradiction.

We conclude that for each \( a \in \mathbb{R} \) and each \( L \in \mathbb{R} \), the statement “\( \lim_{x \to a} f(x) = L \)” is false.

3. Consider the function

\[
g(x) = \begin{cases} 
    x^3, & x \in \mathbb{Q}, \\
    0, & x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

a. (3 points) Prove that \( \lim_{x \to 0} g(x) = 0 \).

**Proof.** You could prove this statement using an \( \epsilon-\delta \) proof. But here is an alternate way. First, note that \( D(g) = \mathbb{R} \), so there certainly exists \( t > 0 \) so that \( \{ x \in \mathbb{R} : 0 < |x - 0| < t \} \subset D(g) \).

Now, observe that if \( |x| < 1 \), then \(-x^2 \leq x^3 \leq x^2 \), and \(-x^2 \leq 0 \leq x^2 \). Let \( g_1(x) = -x^2 \), \( g_2(x) = x^2 \), and \( g_1(x) \leq g(x) \leq g_2(x) \) (if \( x \) is rational, then the inequality becomes \(-x \leq x^3 \leq x \), while if \( x \) is irrational the inequality becomes \(-x \leq 0 \leq x \)). By the limit rule “\( \lim_{x \to a} f(x) = a \)” and the product rule, we have \( \lim_{x \to 0} g_1(x) = 0 \) and \( \lim_{x \to 0} g_2(x) = 0 \). Thus by the squeeze theorem, we have \( \lim_{x \to 0} g(x) = 0 \).

b. (3 points) Prove that for all \( a \in \mathbb{R} \setminus \{0\} \) and all \( L \in \mathbb{R} \), the statement “\( \lim_{x \to a} g(x) = L \)” is false.

(Hint: you can prove this directly, but it might be easier to do a proof by contradiction and use the limit rules discussed in class).

**Proof.** We will do a proof by contradiction. Let \( a \in \mathbb{R} \setminus \{0\} \) and suppose there exists a real number \( L \) so that \( \lim_{x \to a} g(x) = L \). Note as well (by the rule “\( \lim_{x \to a} x = a \)” and the product rule) that \( \lim_{x \to a} x^3 = a^3 \); since \( a \neq 0 \), \( a^3 \neq 0 \). By the quotient rule for limits (discussed in lecture), we have \( \lim_{x \to a} g(x)/x^3 = L/a^3 \). But

\[
g(x)/x = \begin{cases} 
    1, & x \in \mathbb{Q} \setminus \{0\}, \\
    0, & x \in \mathbb{R} \setminus \mathbb{Q} \\
    \text{undefined}, & x = 0
\end{cases}
\]
Thus, if \( f(x) \) is the function from problem 2, then \( f(x) = g(x)/x^3 \) on the open interval \((a/2, 3a/2)\) (if \( a > 0 \)) or \((3a/2, a/2)\) (if \( a < 0 \)); in either case, this open interval contains \( a \). Thus by the “limits are a local property” rule, this implies \( \lim_{x \to a} f(x) = L/a^3 \). But this contradicts 1c, in which we proved that the statement \( \lim_{x \to a} f(x) = L/a^3 \) must be false.

4. (5 points) Let \( f \) and \( g \) be functions, and let \( a \in \mathbb{R} \). Suppose that \( \lim_{x \to a} g(x) = L \) and that \( \lim_{x \to L} f(x) = M \) and \( f(L) = M \). Prove that \( \lim_{x \to a} f \circ g(x) = M \).

**Proof.** Let \( \epsilon > 0 \). Since \( \lim_{x \to L} f(x) = M \), there exists a number \( \delta_1 \) so that for all \( x \in \mathbb{R} \) with \( 0 < |x - L| < \delta_1 \), we have \( |f(x) - M| < \epsilon \). Furthermore, since \( f(L) = M \), we actually have that for all \( x \in \mathbb{R} \) with \( 0 \leq |x - L| < \delta_1 \), we have \( |f(x) - M| < \epsilon \).

Next, since \( \lim_{x \to a} g(x) = L \), there exists a number \( \delta \) so that for all \( x \in \mathbb{R} \) with \( 0 < |x - a| < \delta \), we have \( |g(x) - L| < \delta_1 \) (here, the variable \( \delta_1 \) is taking the role of \( \epsilon \) in the \( \epsilon - \delta \) definition of continuity for \( g \)).

We conclude that if \( x \in \mathbb{R} \) with \( 0 < |x - a| < \delta \), then \( |g(x) - L| < \delta_1 \), and thus \( |f(g(x)) - M| < \epsilon \), i.e. \( |f \circ g(x) - M| < \epsilon \).