Math 120 Homework 2

- Due Friday September 23 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Limits

1. In lecture, we discussed the limit rule: If $\lim_{x\to a} f(x)$ is a real number, and $\lim_{x\to a} g(x)$ is a non-zero real number, then

$$\lim_{x\to a}(f/g)(x) = \big(\lim_{x\to a} f(x)\big)/\big(\lim_{x\to a} g(x)\big).$$

However, if $\lim_{x\to a} g(x) = 0$, then all bets are off. Give an example of functions f, g with $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and:

a (3 points) $\lim_{x\to a} f/g(x) = 1$.

b (3 points) $\lim_{x\to a} f/g(x) = 0$.

c (3 points) For every real number L, the statement "$\lim_{x\to a}(f/g)(x) = L$" is false.

In each of the above problems, prove that your answer is correct.

2. In this problem we will study the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

Thus $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational.

a. (2 points) Prove that if $q/p \in \mathbb{Q}$, $q \neq 0$, and $w \in \mathbb{R}\backslash\mathbb{Q}$, then $w \cdot q/p \in \mathbb{R}\backslash\mathbb{Q}$. Hint: try using a proof by contradiction.

b. (2 points) Prove that if s and t are real numbers with $s < t$, then there exists a number $w \in \mathbb{R}\backslash\mathbb{Q}$ with $s < w < t$. (Hint: HW 1 #6 might be helpful, though you can also prove this fact directly.)

c. (5 points) Prove that for all $a \in \mathbb{R}$ and all $L \in \mathbb{R}$, the statement "$\lim_{x\to a} f(x) = L$" is false.

3. Consider the function

$$g(x) = \begin{cases} x^3, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

a. (3 points) Prove that $\lim_{x\to 0} g(x) = 0$.

b. (3 points) Prove that for all $a \in \mathbb{R}\backslash\{0\}$ and all $L \in \mathbb{R}$, the statement "$\lim_{x\to a} g(x) = L$" is false. (Hint: you can prove this directly, but it might be easier to do a proof by contradiction and use the limit rules discussed in class).

4. (5 points) Let f and g be functions, and let $a \in \mathbb{R}$. Suppose that $\lim_{x\to a} g(x) = L$ and that $\lim_{x\to L} f(x) = M$ and $f(L) = M$. Prove that $\lim_{x\to a} f \circ g(x) = M$.