Math 120 Homework 2

• Due Friday September 23 at start of class.

• If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper.

• Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Limits

1. In lecture, we discussed the limit rule: If \( \lim_{x \to a} f(x) \) is a real number, and \( \lim_{x \to a} g(x) \) is a non-zero real number, then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \left( \lim_{x \to a} f(x) \right) / \left( \lim_{x \to a} g(x) \right).
\]

However, if \( \lim_{x \to a} g(x) = 0 \), then all bets are off. Give an example of functions \( f, g \) with \( \lim_{x \to a} f(x) = 0 \), \( \lim_{x \to a} g(x) = 0 \), and:

a (3 points) \( \lim_{x \to a} f/g(x) = 1 \).

b (3 points) \( \lim_{x \to a} f/g(x) = 0 \).

c (3 points) For every real number \( L \), the statement “\( \lim_{x \to a} \frac{f(x)}{g(x)} = L \)” is false.

In each of the above problems, prove that your answer is correct.

2. In this problem we will study the function

\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \\
0, & x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

Thus \( f(x) = 1 \) if \( x \) is rational, and \( f(x) = 0 \) if \( x \) is irrational.

a. (2 points) Prove that if \( q/p \in \mathbb{Q} \), \( q \neq 0 \), and \( w \in \mathbb{R} \setminus \mathbb{Q} \), then \( w \cdot q/p \in \mathbb{R} \setminus \mathbb{Q} \). Hint: try using a proof by contradiction.

b. (2 points) Prove that if \( s \) and \( t \) are real numbers with \( s < t \), then there exists a number \( w \in \mathbb{R} \setminus \mathbb{Q} \) with \( s < w < t \). (Hint: HW 1 #6 might be helpful, though you can also prove this fact directly.)

c. (5 points) Prove that for all \( a \in \mathbb{R} \) and all \( L \in \mathbb{R} \), the statement “\( \lim_{x \to a} f(x) = L \)” is false.

3. Consider the function

\[
g(x) = \begin{cases} 
x^3, & x \in \mathbb{Q} \\
0, & x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

a. (3 points) Prove that \( \lim_{x \to 0} g(x) = 0 \).
b. (3 points) Prove that for all \( a \in \mathbb{R}\setminus\{0\} \) and all \( L \in \mathbb{R} \), the statement \( \lim_{x \to a} g(x) = L \) is false. (Hint: you can prove this directly, but it might be easier to do a proof by contradiction and use the limit rules discussed in class).

4. (5 points) Let \( f \) and \( g \) be functions, and let \( a \in \mathbb{R} \). Suppose that \( \lim_{x \to a} g(x) = L \) and that \( \lim_{x \to L} f(x) = M \) and \( f(L) = M \). Prove that \( \lim_{x \to a} f \circ g(x) = M \).