American Math. Soc.
85 (1978) no. 4, p. 246 – p. 256

WHEN IS A FUNCTION THAT SATISFIES THE CAUCHY–RIEMANN EQUATIONS ANALYTIC?

J. D. GRAY AND S. A. MORRIS

1. The Looman–Menchoff theorem—An extension of Goursat's theorem. It is well known[1] that a complex-valued function $f = u + iv$, defined and analytic on a domain D in the complex plane satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D. The standard textbooks, such as those authored by Ahlfors, Cartan, Churchill, Jameson, Knopp, Sansone and Gerretson, avoid answering the question as to whether or not the converse holds. Most instead offer the following partial converse due to Goursat [13].

THEOREM 1. *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in* D,
(ii) u, v *satisfy the Cauchy–Riemann equations everywhere in* D, *and if further*
(iii) f *is continuous in* D,
(iv) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *are continuous in* D,
then f is analytic in D.

This is a substantially revised version of an article by the present authors and S. A. R. Disney that appeared in the Gazette of the Australian Mathematical Society 2 (3) (1975), 67–81. S. A. R. Disney's name does not appear above only because he preferred it that way.

[1] This is a rare instance of a well-known result that is indeed well known.

The remaining standard texts offer the stronger result:

THEOREM 2. *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *u, v, as functions of two real variables, are differentiable everywhere in D,*
then f is analytic in D.

Recently the authors began a search to discover precisely what is known regarding the converse. The only modern book we were able to find tnat addresses itself to this problem is Derrick [8]. He points out that far weaker conditions than those of Theorem 2 are known to imply analyticity but that the Cauchy–Riemann equations themselves do not imply analyticity! Indeed, the function f given by

$$f(z)=\begin{cases}\exp(-z^{-4}) & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

first noticed by Looman [20, 107], (see also [39, 70]), is readily seen to satisfy the Cauchy–Riemann equations everywhere, but, as $f(z)/z\to\infty$ as $z\to 0$ with $\arg z=\pi/4$, fails to be analytic at the origin. Observe that f must have an essential singularity at 0 otherwise $\partial f/\partial x$ could not exist there.

Derrick [8] suggests that the 'best' result in this direction appears to be

THEOREM 3. (Looman–Menchoff) *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*
(iii) *f is continuous in D,*
then f is analytic in D.

Menchoff's proof (see [34, 199] and [24, 9]), based on the concepts of Lebesgue integration and Baire category is, according to Saks [36] "... undoubtedly one of the most elegant and unexpected applications of the modern theory of real functions to the elementary problems of an entirely classical aspect." A proof of the theorem is given in the appendix.

A word of caution. The naive local version of the Looman–Menchoff theorem is: if a function is continuous at z_0 and satisfies the Cauchy–Riemann equations there, it is complex-differentiable at z_0. This assertion is false! For example [8, 15], the function

$$f(z)=\begin{cases}z^5/|z|^4 & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

is continuous everywhere, satisfies the Cauchy–Riemann equations at 0, but is not complex-differentiable at the origin. To the best of our knowledge the strongest result in this direction is the standard one: if $f=u+iv$ is such that (i) u, v are differentiable at z_0, (ii) u, v satisfy the Cauchy–Riemann equations at z_0, then f is (complex) differentiable at z_0. See [17, 35].

Although Looman and Menchoff clearly did improve on Goursat's theorem others have obtained still more subtle results.

2. Extensions of the theorems of Green, Morera and Goursat. The earliest contribution to the problem appears to be that of Paul Montel who, in a 1913 note in the *Comptes Rendus.* asserted the

THEOREM 4. *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D. and if further*
(iii) *f is bounded in D,*
then f is analytic in D.

Recall that a function f on D is said to be locally bounded if it is bounded in some neighbourhood