[10] 1. a) Determine where the function $f(x+iy) = e^x + ie^{2y}$ is differentiable *and* where it is analytic.

b) Let $f(z)$ be entire. Prove that $\overline{f(\bar{z})}$ is also entire.

1

a) $u(x,y) = e^x$, $v(x,y) = e^{2y}$

$u_x = e^x$ $\quad v_x = 0$

$u_y = 0$ $\quad v_y = 2e^{2y}$

C-R Eqns: $u_x = v_y$ iff $e^x = 2e^{2y}$

iff $e^x = e^{\ln(2)+2y}$

iff $x = \ln(2) + 2y$

$u_y = -v_x$ always

Since partial derivatives of u and v exist and are continuous everywhere,

$f(x+iy)$ is differentiable at $z_0 = x_0 + iy_0$

iff

$x_0 = \ln(2) + 2y_0$

Thus, f is differentiable only at points on a straight line. Thus, there is no point such that f is differentiable on an open disk centered at the point.

Thus, f is analytic nowhere.

b) Write $f(z) = u(x,y) + iv(x,y)$

Write $\overline{f(\bar{z})} = a(x,y) + ib(x,y)$

where $a(x,y) = u(x,-y)$, $b(x,y) = -v(x,-y)$

$a_x(x,y) = u_x(x,-y)$ $\quad b_x(x,y) = -v_x(x,-y)$

$a_y(x,y) = -u_y(x,-y)$ $\quad b_y(x,y) = v_y(x,-y)$

Thus, the C-R eqns. for $\overline{f(\bar{z})}$ become

$$u_x(x,-y) = v_y(x,-y)$$

$$-u_y(x,-y) = v_x(x,-y)$$

But these follow from the C-R eqns. for $f(z)$. Since an entire function has partial derivative of all orders, u_x, u_y, v_x & v_y all exist and are continuous. Thus, the same can be said for a_x, a_y, b_x & b_y.

Thus $\overline{f(\bar{z})}$ satisfies C-R and ~~[illegible]~~ all partial derivatives exist and are continuous. (everywhere!)

Thus, $\overline{f(\bar{z})}$ is entire

[10] 2. Determine the domain of analyticity of:

a) $f(z) = \frac{1}{e^z+2}$

b) $f(z)$=Log(Log$(z) - i\frac{\pi}{2})$

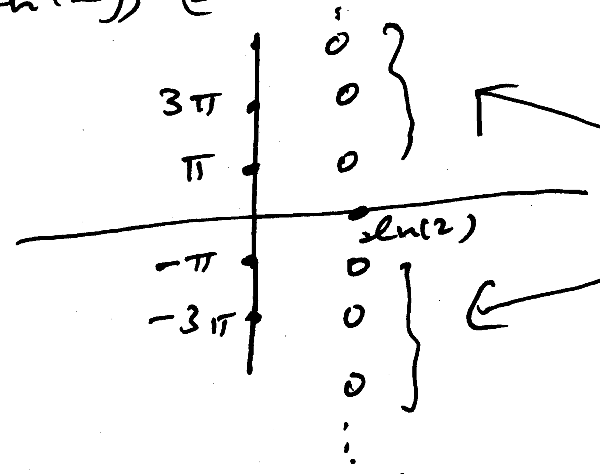
f is analytic at z iff $e^z + 2 \neq 0$

iff $e^z \neq -2$

iff $e^{x+iy} \neq e^{\ln(2) + i((2k+1)\pi)}$

So, f is analytic at z iff

$z \notin \{(\ln(2), (2k+1)\pi) : k \in \mathbb{Z}\}$



z not in

b) f is analytic at z_0 iff

(i) $z_0 \notin \{x+iy : y=0, x \leq 0\}$

and (ii) $\text{Log}(z_0) - i\pi/2 \notin \{x+iy : y=0, x \leq 0\}$

(ii) is equivalent to:

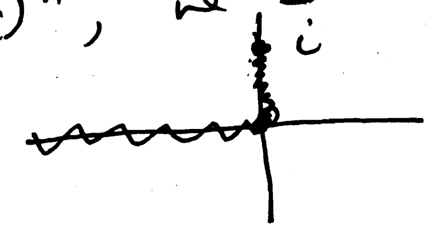
(ii)' $\text{Log}(z_0) \notin \{x+iy : y = \frac{\pi}{2}, x \leq 0\}$

Since $\text{Log}(z_0) = \text{Log}|z_0| + i\,\text{Arg}(z_0)$, (ii)' is

equivalent to: $z_0 \notin \{z : |z| \leq 1, \text{Arg}(z) = \frac{\pi}{2}\}$ (ii)''

Combining (i) and (ii)'', we see that f is

analytic for $z_0 \notin$



[10] 3. a) Find the first three coefficients, a_0, a_1, a_2, in the Taylor series of $\frac{1}{1+e^z}$ around $z_0 = 0$. What is the radius of convergence? (give a reason for your answer)

b) Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of some analytic $f(z)$ at 0. Show that if f is even (i.e., $f(z) = f(-z)$ for all z), then $a_n = 0$ for all odd n.

(a) $f(z) = \frac{1}{1+e^z}$ $\therefore$ $\underline{a_0 = \frac{1}{2}}$

$f'(z) = \frac{-e^z}{(1+e^z)^2}$ $\therefore$ $\underline{a_1 = -\frac{1}{4}}$

$f''(z) =$ ... after some simplification..

$\frac{-e^z(1+e^z)+2e^{2z}}{(1+e^z)^3}$ $\therefore$ $\underline{a_2 = 0}$

The Taylor series converges on the largest disk, centered at 0, on which $f(z)$ is analytic.
Now, f is analytic at z iff $e^z \neq -1$
iff $z \neq ((2k+1)\pi)i \quad k \in \mathbb{Z}$

Thus the Taylor series converges on $\{|z| < \pi\}$. It cannot converge on any larger disk since as $z \to \pi i$, $f(z) \to \infty$. Thus, $R = \pi$.

(b) $f(z) = f(-z) \Rightarrow \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_j (-1)^j z^j$

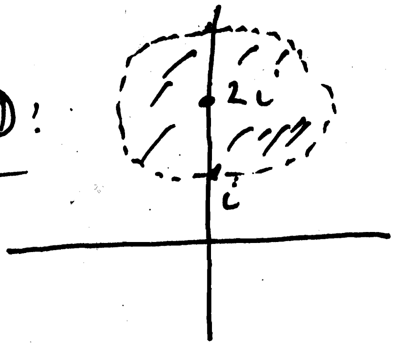
$\Rightarrow \sum_{\substack{j=0 \\ \{j \text{ odd}\}}}^{\infty} 2a_j z^j = 0$

By uniqueness of Taylor series (for the 0 function)
$2a_j = 0$ for all odd j. Thus, $a_j = 0$ for all odd j.

[10] 4. Find two different Laurent series for the function $f(z) = Log(z) + 1/(z - i)$ in annuli centered at $z_0 = 2i$ and specify the domain of convergence for each series.

Let $D = \{ |z-2i| < 1 \}$

D:



Since $i \notin D$ and D does not intersect the non-positive x-axis, f is analytic on the disk D.
Thus, the Laurent series for f on D is the same as its Taylor series on D.

Write $g(z) = Log(z)$ and $h(z) = 1/(z-i)$

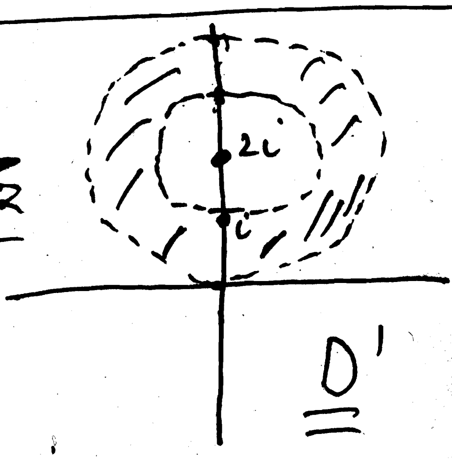
$(j \geq 1)$ $\quad g^{(j)}(z) = \dfrac{(-1)^{j-1}(j-1)!}{z^j} \qquad h^{(j)}(z) = \dfrac{(-1)^j j!}{(z-i)^{j+1}}$

Plug in $z = 2i$ and divide by $j!$:

$$f(z) = \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j(2i)^j} (z-2i)^j \right) + Log(2i) + \sum_{j=0}^{\infty} \frac{(-1)^j}{i^{(j+1)}} (z-2i)^j$$

Let $D' = \{ 1 < |z-2i| < 2 \}$

Since $i \notin D'$ and D' does not intersect the non-positive x-axis, f is analytic on the annulus D'.



D'

The Taylor series for $g(z) = \mathrm{Log}(z)$ converges in the disk $D'' = \{|z-2i| < 2\}$ (since $\mathrm{Log}(z)$ is analytic in that disk). Thus, the Laurent series for $g(z)$ is the same as its Taylor series on D' and thus on D' we have (from the above)

$$g(z) = \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j(2i)^j}(z-2i)^j\right) + \mathrm{Log}(2i) \qquad (*)$$

However, $h(z) = 1/(z-i)$ is not analytic on D'' and so we must resort to other means to find its Laurent series in D'. Using the "geometric series trick," we see

$$h(z) = \frac{1}{z-i} = \frac{1}{(z-2i)+i} = \frac{1}{(z-2i)}\left(\frac{1}{1+\left(\frac{i}{z-2i}\right)}\right)$$

$$= \left(\frac{1}{z-2i}\right)\sum_{j=0}^{\infty}\left(\frac{-i}{z-2i}\right)^j = \sum_{j=1}^{\infty}\frac{(-i)^{j-1}}{(z-2i)^j} \qquad (**)$$

and this converges in D' since $\left|\frac{(-i)}{z-2i}\right| = \frac{1}{|z-2i|} < 1$

Thus, on D' ~~we~~ we have:

$$f(z) = (*) + (**)$$

[20] 5. Calculate each of the following contour integrals (assume every closed contour is oriented counter-clockwise)

a) $\int_C \bar{z}^2 dz$ where C the line segment from 0 to $1+2i$.

b) $\int_C \frac{Log(z)}{z^2+9} dz$ where C is the circle $|z-4i| = 3$.

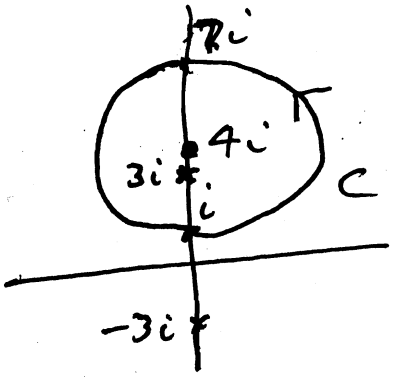
c) $\int_C \sin(3/z) dz$ where C is the unit circle $|z| = 1$.

d) $\int_C \frac{1}{(z-\alpha)(z-\beta)} dz$ where C is a counter clockwise circle and α and β lie strictly inside C.

a) $$\int_C \bar{z}^2 dz = \int_0^1 (t(1-2i))^2 (1+2i)\, dt$$

$$= (1-2i)^2(1+2i) \int_0^1 t^2 dt = \frac{5}{3}(1-2i)$$

b)



Singularities at $\pm 3i$ and on non-positive x-axis. But of these, the only singularity inside the contour is $3i$

$$\int_C \frac{Log(z)}{z^2+9} dz = (2\pi i)(Res(3i))$$

$$\frac{Log(z)}{z^2+9} = \frac{g(z)}{z-3i}$$ where $g(z) = \frac{Log(z)}{z+3i}$ and $g(z)$ is analytic and non-zero at $z_0 = 3i$

Thus, $Res(3i) = g(3i) = \frac{Log(3i)}{6i}$

$$\therefore \int_C \frac{Log(z)}{z^2+9} dz = \frac{\pi}{3} Log(3i) = \frac{\pi}{3}\left(Log(3) + \frac{i\pi}{2}\right)$$

(by Cauchy Residue Theorem)

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c) Since $\sin(w)$ is entire, $f(z) = \sin(3/z)$ is analytic everywhere except $z_0 = 0$, which is inside the unit circle.

Thus, $\int_C \sin(3/z)\,dz = 2\pi i \,\mathrm{Res}(0)$

$$\sin(3/z) = \frac{3}{z} - \frac{\left(\frac{3}{z}\right)^3}{3!} + \frac{\left(\frac{3}{z}\right)^5}{5!} - \cdots$$

$$= 3z^{-1} - \frac{3^3}{3!}z^{-3} + \frac{3^5}{5!}z^{-5} - \cdots$$

$\therefore \mathrm{Res}(0) = 3$ $\quad \therefore \int_C \sin(3/z)\,dz = \underline{6\pi i}$

d) <u>Case 1:</u> <u>$\alpha = \beta$</u>. Since $\alpha = \beta$ lies inside the circle, the function $f(z) = \frac{1}{(z-\alpha)^2}$ has an anti-derivative $F(z) = \frac{-1}{(z-\alpha)}$ on a domain containing C. Thus, by the Fundamental Theorem of contour integrals, $\int_C f(z)\,dz = 0$.

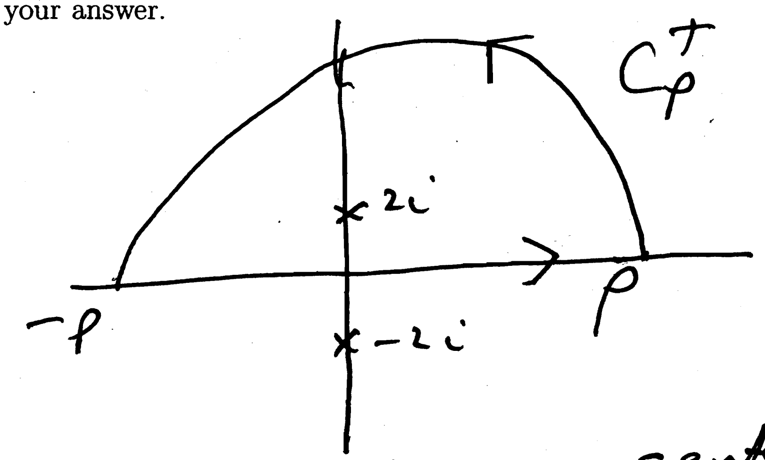
<u>Case 2:</u> $\alpha \neq \beta$. Then $f(z) = \frac{1}{(z-\alpha)(z-\beta)}$ has simple poles at α and β. $\mathrm{Res}(f,\alpha) = \frac{1}{\alpha-\beta}$, $\mathrm{Res}(f,\beta) = \frac{1}{\beta-\alpha}$

$$\therefore \int_C \frac{1}{(z-\alpha)(z-\beta)}\,dz = 2\pi i\left(\frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha}\right) = 0.$$

[10] 6. Find

$$\int_0^\infty \frac{x^2+1}{(x^2+4)^2}dx$$

and justify your answer.



Sorry for the poorly drawn semi-circle

C_ρ^+: semi-circle of radius ρ, centered at 0 (pos. oriented)

Let $\Gamma_\rho = [-\rho, \rho] \cup C_\rho^+$

Let $f(z) = \dfrac{z^2+1}{(z^2+4)^2} = \dfrac{\overbrace{\left.\dfrac{z^2+1}{(z+2i)^2}\right\}}^{g(z)}}{(z-2i)^2}$ $\therefore$ a pole of order 2

where $g(z)$ is analytic and non-zero at $z_0 = 2i$

$\therefore \operatorname{Res}(2i) = g'(2i) =$ (after some computation) $\dfrac{11}{32i}$

$\therefore \displaystyle\int_{\Gamma_\rho} f(z)dz = 2\pi i\left(\frac{11}{32i}\right) = \frac{11}{16}\pi$ $(*)$

Since degree $(z^2+4)^2 \geq 2 +$ degree (z^2+1),

$\displaystyle\lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\,dz = 0.$ $(**)$

~~Then~~ And $\displaystyle\lim_{\rho\to\infty} \int_{\Gamma_\rho} f(z)dz = \lim_{\rho\to\infty}\left(\int_{-\rho}^{\rho} \frac{x^2+1}{(x^2+4)^2}dx + \int_{C_\rho^+} f(z)dz\right)$ $(***)$

Comparing (*) and (**) and (***), we see

$$\int_{-\infty}^{\infty} \frac{x^2+1}{(x^2+4)^2}\,dx = \frac{11}{16}\pi$$

By symmetry,

$$\int_{0}^{\infty} \frac{x^2+1}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2+1}{(x^2+4)^2}\,dx$$

$$= \frac{11}{32}\pi$$

[10] 7. In each case below, find an example of a function $f(z)$ satisfying:

a) The Taylor series at $z = 3$ has radius of convergence $= 5$, and $f(2) = 2$.

b) $f(z)$ has an essential singularity at $z = 0$, a pole of order 1 at $z = 1$, a pole of order 2 at $z = 2$ and is analytic everywhere else.

c) $\int_C f(z)dz$ is equal to i when C is the circle $|z| = 1$, and it is equal to 1 when C is the circle $|z| = 3$.

a) Let $f(z) = \boxed{\dfrac{-12}{z-8}}$

$$f(2) = \frac{-12}{2-8} = 2$$

The Taylor series converges on the largest disk centered at 3 on which $f(z)$ is analytic. Since $f(z)$ is analytic everywhere except at $z_0 = 8$, the Taylor series converges on $\{|z-3| < 5\}$. It cannot converge on any larger disk since $\lim_{z \to 8} f(z) = \infty$. Thus, the Taylor series has radius of convergence $= 5$.

b) $$f(z) = \frac{e^{1/z}}{(z-1)(z-2)^2}$$

$z_0 = 1$ is pole of order 1: because $f(z) = \dfrac{\overbrace{\frac{e^{1/z}}{(z-2)^2}}^{g(z)}}{(z-1)}$ analytic and nonzero at 1

$z_0 = 2$ is pole of order 2: because $f(z) = \dfrac{\overbrace{\frac{e^{1/z}}{z-1}}^{g(z)}}{(z-2)^2}$

$z_0 = 0$ is essential singularity; ~~because~~ First note that $e^{1/z}$ has essential singularity at 0

since $e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j!} z^{-j}$

Thus, given $b \in \mathbb{C}$, $\exists$ ~~sequence~~ $z_n \to 0$ s.t.

$$\lim_{z_n \to 0} e^{1/z} = 4b$$

Thus,

$$\lim_{z_n \to 0} \frac{e^{1/z}}{(z-1)(z-2)^2} \to \frac{4b}{4} = b$$

Thus, 0 is an essential singularity for $f(z)$.

Note: Perhaps a simpler example for b) is $e^{1/z} + 1/(z-1) + 1/(z-1)^2$

c) $f(z) = \dfrac{1}{2\pi z} + \dfrac{(1-i)/2\pi i}{z-2}$

When $C = \{|z|=1\}$

$$\int_C f(z)\,dz = \int_C \frac{1}{2\pi z}\,dz + \int_C \frac{(1-i)/2\pi i}{z-2}\,dz$$

$$= 2\pi i \, \mathrm{Res}\left(\tfrac{1}{2\pi z}; 0\right) + 0$$

since $\frac{1}{z-2}$ is analytic inside and on

$$= 2\pi i\left(\frac{1}{2\pi}\right)$$

$$= i$$

When $C = \{|z| = 3\}$

$$\int_C f(z)\,dz = \int_C \frac{1}{2\pi z}\,dz + \int_C \frac{(1-i)/2\pi i}{z-2}\,dz$$

$$= 2\pi i\left(\mathrm{Res}\left(\tfrac{1}{2\pi z}; 0\right)\right) + 2\pi i\left(\mathrm{Res}\, \frac{(1-i)/2\pi i}{z-2}; 2\right) = \left.\begin{matrix} \frac{2\pi i}{2\pi} \\ + \\ \frac{2\pi i(1-i)}{2\pi i} \end{matrix}\right\} = i + 1 - i = 1$$

[10] 8. a) Let $f(z)$ be an entire function satisfying

$$|f(z)| \geq 1$$

for all z. Show that $f(z)$ must be a constant function.

b) Let $f(z)$ be an entire function whose image lies above the parabola $y = x^2$ in plane, i.e., f maps the complex plane into $\{(x, y) : y > x^2\}$. Show that $f(z)$ must be a constant function.

a) Let $g(z) = \dfrac{1}{f(z)}$

Since $f(z)$ is entire and never 0

(since $|f(z)| \geq 1$), $g(z)$ is entire.

And $|g(z)| = \left|\dfrac{1}{f(z)}\right| \leq 1$

So, $g(z)$ is a bounded, entire function.

Thus, $g(z)$ is constant. Thus, $f(z)$ is constant.

b) Let $h(z) = f(z) + i$

Then $h(z)$ is entire and the image of $h(z)$ lies above the parabola $\{(x,y) : y > x^2 + 1\}$

If $w = (x,y)$ is in the image of $h(z)$, then

$|w| = \sqrt{x^2 + (x^2+1)^2} \geq 1$.

Thus, ~~[illegible]~~ $|h(z)| \geq 1$.

Apply part a)