

Solutions

1. Give all possible values of the following in the form $x + iy$.

(a) $(1 - i)^i$

Solution:

$$\begin{aligned}(1 - i)^i &= e^{i \log(1-i)} = e^{i(\operatorname{Log}|1-i| + i \operatorname{arg}(1-i))} = e^{i(\operatorname{Log}\sqrt{2} + i(-\frac{\pi}{4} + 2k\pi))} \\ &= e^{\frac{\pi}{4} - 2k\pi + i \operatorname{Log}\sqrt{2}} = e^{\frac{\pi}{4} - 2k\pi} e^{i \operatorname{Log}\sqrt{2}} = e^{\frac{\pi}{4} - 2k\pi} (\cos \operatorname{Log}\sqrt{2} + i \sin \operatorname{Log}\sqrt{2}) \\ &= e^{\frac{\pi}{4} - 2k\pi} \cos \operatorname{Log}\sqrt{2} + i e^{\frac{\pi}{4} - 2k\pi} \sin \operatorname{Log}\sqrt{2} \quad \text{for any integer } k.\end{aligned}$$

(b) $\operatorname{Log} e^{3 + \frac{5}{2}\pi i}$

Solution:

$$\begin{aligned}\operatorname{Log} e^{3 + \frac{5}{2}\pi i} &= \operatorname{Log}(e^3 e^{\frac{5}{2}\pi i}) = \operatorname{Log}(ie^3) = \operatorname{Log}|ie^3| + i \operatorname{Arg}(ie^3) \\ &= \operatorname{Log}(e^3) + i \frac{\pi}{2} = 3 + i \frac{\pi}{2}.\end{aligned}$$

(c) $\cos(\frac{\pi}{2} + i)$

Solution:

$$\begin{aligned}\cos(\frac{\pi}{2} + i) &= \frac{e^{i(\frac{\pi}{2} + i)} + e^{-i(\frac{\pi}{2} + i)}}{2} = \frac{e^{-1 + i\frac{\pi}{2}} + e^{-1 - i\frac{\pi}{2}}}{2} = \frac{e^{-1} e^{i\frac{\pi}{2}} + e^{-1} e^{-i\frac{\pi}{2}}}{2} \\ &= \frac{\frac{1}{e} \cdot i + e \cdot (-i)}{2} = i \left(\frac{1}{2e} - \frac{e}{2} \right).\end{aligned}$$

2. Find all values of z in \mathbb{C} where $f(z) = x^3 + iy^3$ is

(a) differentiable

Solution: In order for f to be differentiable at any z , the Cauchy-Riemann equations must be true at z . For this problem, $u = x^3$ and $v = y^3$, so

$$\begin{aligned}u_x &= 3x^2 \\u_y &= 0 \\v_x &= 0 \\v_y &= 3y^2.\end{aligned}$$

The Cauchy-Riemann equations then say that $u_x = v_y$, or $3x^2 = 3y^2$, and $u_y = -v_x$, or $0 = -0$. The Cauchy-Riemann equations are then satisfied exactly when $3x^2 = 3y^2$, which happens when $x^2 = y^2$ or $x = \pm y$. So, these are the only values of z for which f could possibly be differentiable. To know that f is actually differentiable, we also must check that the partial derivatives u_x , u_y , v_x , and v_y are all continuous at these points. But these partials are polynomials and so continuous everywhere. Therefore, f is differentiable precisely at those values of z for which $x = \pm y$.

(b) analytic

Solution: In order for f to be analytic at a point z , there must be a disk centered at z in which f is differentiable at every point. Since the set of points where f is differentiable is just a union of two lines, there is no disk at all in which f is differentiable at every point, and so f is not analytic at any point of \mathbb{C} .

3. Find a function f analytic in the entire complex plane whose imaginary part is $v(x, y) = y^3 - 3x^2y + 4xy - x$ and for which $f(1 + i) = 2 + i$.

Solution: If $f = u + iv$ is such a function, then u and v need to satisfy the Cauchy-Riemann equations. This means that u is a function for which

$$u_x = v_y = 3y^2 - 3x^2 + 4x \text{ and } u_y = -v_x = -(-6xy + 4y - 1) = 6xy - 4y + 1.$$

We can use these to solve for u . Since $u_x = 3y^2 - 3x^2 + 4x$,
 $u = \int (3y^2 - 3x^2 + 4x) dx = 3xy^2 - x^3 + 2x^2 + g(y)$ for some function $g(y)$.

Then $u_y = 6xy + g'(y)$, and so we know that $g'(y) = -4y + 1$. This means that $g = \int (-4y + 1) dy = -2y^2 + y + C$ for some constant C . Putting these together, we get

$$u = 3xy^2 - x^3 + 2x^2 - 2y^2 + y + C.$$

To solve for C , we use the fact that $f(1 + i) = 2 + i$. $f(1 + i) = u(1, 1) + iv(1, 1) = (3 - 1 + 2 - 2 + 1 + C) + i(1 - 3 + 4 - 1) = (3 + C) + i$. This means that $3 + C = 2$, and so $C = -1$. We can now write our function f :

$$f = u + iv = (3xy^2 - x^3 + 2x^2 - 2y^2 + y - 1) + i(y^3 - 3x^2y + 4xy - x).$$

To see that f is analytic in the entire complex plane, notice that it satisfies the Cauchy-Riemann equations everywhere, and so since all of the partials u_x , u_y , v_x , and v_y are continuous in \mathbb{C} , f is differentiable everywhere in the complex plane and therefore analytic in the entire complex plane.

In fact, it turns out that $f(z) = -z^3 + 2z^2 - iz - 1$.

4. (a) Find the domain of analyticity of $f(z) = \sqrt{(\text{Log } z) - \frac{\pi}{2}i}$, where the square root is given by its principal branch.

Solution: There are two types of points at which f can be nonanalytic. Firstly, if z_0 is on the negative real axis, then the function $\text{Log } z$ is nonanalytic at $z = z_0$, and so certainly f would be nonanalytic at $z = z_0$ as well. Secondly, if $(\text{Log } z_0) - \frac{\pi}{2}i$ is on the negative real axis, then the function \sqrt{z} is nonanalytic at $z = (\text{Log } z_0) - \frac{\pi}{2}i$, and so $f(z)$ is nonanalytic at $z = z_0$.

Our goal then is to describe the points z_0 for which either z_0 is on the negative real axis (this is already done), or $(\text{Log } z_0) - \frac{\pi}{2}i$ is on the negative real axis. Note that

$$(\text{Log } z_0) - \frac{\pi}{2}i = \text{Log } |z_0| + i \text{Arg } z_0 - \frac{\pi}{2}i = \text{Log } |z_0| + i \left(\text{Arg } z_0 - \frac{\pi}{2} \right).$$

If this number is on the negative real axis, then $\text{Log } |z_0| \leq 0$ and $\text{Arg } z_0 - \frac{\pi}{2} = 0$.

$\text{Log } |z_0| \leq 0 \Leftrightarrow 0 < |z_0| \leq 1$, and $\text{Arg } z_0 - \frac{\pi}{2} = 0 \Leftrightarrow \text{Arg } z_0 = \frac{\pi}{2}$. So, the set of z_0 that we are looking for is the set of complex numbers with modulus in $(0, 1]$ and argument $\frac{\pi}{2}$, which is just the interval $\{z \in \mathbb{C} : x = 0, y \in (0, 1]\}$.

The domain of analyticity of f is then all values of z that are not on the negative real axis $\{z \in \mathbb{C} : y = 0, x \leq 0\}$ or the interval $\{z \in \mathbb{C} : x = 0, y \in (0, 1]\}$.

(b) Find $f(1)$.

Solution:

$$\begin{aligned} f(1) &= \sqrt{(\text{Log } 1) - \frac{\pi}{2}i} = \sqrt{0 - \frac{\pi}{2}i} = e^{\frac{1}{2}\text{Log}(-\frac{\pi}{2}i)} = e^{\frac{1}{2}[\text{Log}|\frac{\pi}{2}i| + i\text{Arg}(-\frac{\pi}{2}i)]} \\ &= e^{\frac{1}{2}[\text{Log } \frac{\pi}{2} + i(-\frac{\pi}{2})]} = e^{\frac{1}{2}\text{Log } \frac{\pi}{2} + i(-\frac{\pi}{4})} = e^{\frac{1}{2}\text{Log } \frac{\pi}{2}} e^{i(-\frac{\pi}{4})} \\ &= e^{\frac{1}{2}\text{Log } \frac{\pi}{2}} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) = e^{\frac{1}{2}\text{Log } \frac{\pi}{2}} \frac{\sqrt{2}}{2} - i e^{\frac{1}{2}\text{Log } \frac{\pi}{2}} \frac{\sqrt{2}}{2}. \end{aligned}$$