Problem 1. [10 points]

(1) Let $z = \left(\frac{1-i}{\sqrt{2}}\right)^{2010}$. Find $|z|$ and $\operatorname{Arg} z$.

$$\left|\frac{1-i}{\sqrt{2}}\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1.$$

$$\begin{aligned} \operatorname{Arg} z &= \operatorname{Arg} e^{-\frac{\pi}{4}i \cdot 2010} \\ &= \operatorname{Arg} e^{-\left(502\pi + \frac{\pi}{2}\right)i} \\ &= -\frac{\pi}{2} \end{aligned}$$

(2) Find all distinct values of $(-8)^{\frac{1}{3}}$. The answer should be in the form $x + iy$ and simplified.

$$(-8)^{\frac{1}{3}} = 2\, e^{\frac{\pi i + 2k\pi i}{3}}, \qquad k \in \mathbb{Z}.$$

$k=0$, $\quad 2e^{\frac{\pi}{3}i} = 2\cdot \cos\frac{\pi}{3} + i \cdot 2\sin\frac{\pi}{3} = 1+\sqrt{3}\,i$

$k=1$, $\quad 2e^{\frac{3\pi i}{3}} = 2\cos\pi + i2\sin\pi = -2$

$k=2$. $\quad 2e^{\frac{5\pi i}{3}} = 2\cos\frac{5\pi}{3} + i\,2\cdot\sin\frac{5\pi}{3} = 2\cos\left(-\frac{\pi}{3}\right) + i\,2\sin\left(-\frac{\pi}{3}\right)$

$= 1 - \sqrt{3}\,i$

Problem 2. **[10 points]** Let $u(x, y) = e^x \sin y + e^{-x} \cos y$ for all real x and y.

(a) Verify that $u(x, y)$ is a harmonic function.

$$u_x = e^x \sin y - e^{-x} \cos y$$
$$u_y = e^x \cos y - e^{-x} \sin y$$

$$u_{xx} = e^x \sin y + e^{-x} \cos y$$
$$u_{yy} = -e^x \sin y - e^{-x} \cos y$$

$$\therefore \quad u_{xx} + u_{yy} = 0$$

(b) Find the entire function $f(z)$ whose real part is the function $u(x, y)$ and $f(i\pi) = -1 + i$.

Set $f = u + iv$

CR equations need to be satisfied:

$$v_x = -u_y = -e^x \cos y + e^{-x} \sin y$$

$$\therefore v = \int \left(-e^x \cos y + e^{-x} \sin y \right) dx$$

$$= -e^x \cos y - e^{-x} \sin y + \varphi(y)$$

$$\therefore v_y = e^x \sin y - e^{-x} \cos y + \varphi'(y)$$

$$v_y = u_x = e^x \sin y - e^{-x} \cos y$$

$$\therefore \quad \varphi'(y) = 0 \qquad \therefore \varphi(y) = C.$$

$$\therefore f(z) = \left(e^x \sin y + e^{-x} \cos y \right) + i\left(-e^x \cos y - e^{-x} \sin y + C \right)$$

$$-1 + i = f(i\pi) = \left(e^0 \sin \pi + e^{-0} \cos \pi \right) + i\left(-e^0 \cos \pi - e^{-0} \sin \pi + C \right)$$
$$= -1 + i(1 + C)$$

$$\therefore \quad C = 0$$

$$\therefore f(z) = \left(e^x \sin y + e^{-x} \cos y \right) + i\left(-e^x \cos y - e^{-x} \sin y \right)$$

Problem 3. **[10 points]** Let $f(z)$ be the function $\bar{z}^2 + 3z$.

(a) Find all z where f is (complex) differentiable.

$$\begin{aligned} \bar{z}^2 + 3z &= \left(\overline{x+iy}\right)^2 + 3(x+iy) \\ &= (x-iy)^2 + 3x + 3iy \\ &= \underbrace{(x^2-y^2+3x)}_{u} + i\underbrace{(-2xy+3y)}_{v} \end{aligned}$$

$u_x = 2x+3$, $u_y = -2y$

$v_x = -2y$, $v_y = -2x+3$, u_x, u_y, v_x, v_y continuous

CR is satisfied when

$u_x = v_y \Rightarrow 2x+3 = -2x+3 \Rightarrow x=0$

$u_y = -v_x \Rightarrow -2y = 2y \Rightarrow y=0$

$\therefore$ f is differentiable at $z=0$

(b) Determine where f is analytic.

nowhere analytic.

Analytic at a point means the function is (complex) differentiable in a neighborhood of the point.

f is only differentiable at one point : $z=0$.

Problem 4. **[10 points]** Let $f(z)$ be the principal branch of $\left(z^2+iz+2\right)^{\frac{1}{2}}$. Determine the domain of analyticity of $f(z)$ (namely, *all* values of z where f is analytic).

$$f(z) = e^{\frac{1}{2}\operatorname{Log}(z^2+iz+2)}$$

Set $w = z^2 + iz + 2$. The cut for $\operatorname{Log} w$ is along $w \le 0$. $\Rightarrow$ $z^2 + iz + 2 \le 0$

$$\Rightarrow (x+iy)^2 + i(x+iy) + 2 \le 0$$

$$\Rightarrow (x^2 - y^2 - y + 2) + i(2xy + x) \le 0$$

$$\Rightarrow \begin{cases} x^2 - y^2 - y + 2 \le 0 & (1) \\ 2xy + x = 0 & (2) \end{cases}$$

$(2) \Rightarrow x = 0$ or $y = -\frac{1}{2}$

if $x = 0$, $(1) \Rightarrow -y^2 - y + 2 \le 0$

$$\Rightarrow y^2 + y - 2 \geq 0 \Rightarrow (y-1)(y+2) \ge 0$$

$$\Rightarrow y \ge 1 \text{ or } y \le -2$$

If $y = -\frac{1}{2}$, $(1) \Rightarrow x^2 - \frac{1}{4} + \frac{1}{2} + 2 \le 0$

$\Rightarrow x^2 + 2 + \frac{1}{4} \le 0$, no solution

$\therefore$ Domain of analyticity of $f(z)$ is

$$\mathbb{C} \setminus \{ yi : y \ge 1 \text{ or } y \le -2 \}$$