# Math 300 Course Review:

The Final Exam will be held on Monday, April 26th, 12:00 p.m. - 2:30 p.m. in SWNG 121. Calculators, laptops, notes, "cheat sheets," etc. will NOT be allowed.

The Final Exam will cover all material from the course, including the following sections of the text: 1.1-1.6, 2.1-2.5, 3.1-3.3, 3.5, 4.1-4.6, 5.1-5.3, 5,5, 5.6, 6.1, 6.3. However, the Final Exam will emphasize material in chapters 4, 5 and 6. But you can expect at least one of the problems to be based on midterm material, and of similar difficulty to one of the more difficult problems on the midterm.

The best ways to prepare for the Final Exam are to review the homework and midterm problems and solutions, your class notes, text material, and work on the past final exams posted on the course website and other problems in the text. Below is a review of some the main ideas and key points to remember. You may also want to review the summary sections at end of each chapter in the text.

## 1. BASICS OF COMPLEX NUMBERS: Chapter 1

The set of complex numbers is the set $\mathbb{C} = \{a + ib\}$ where $a, b$ are real numbers. We defined complex addition and multiplication on this set, under which the basic rules of arithmetic were obeyed. We viewed $\mathbb{C}$ as the 2 dimensional plane, and this gave a geometric context to the complex numbers, and to complex arithmetic in particular.

(1) Know the definition of complex numbers, and their cartesian and polar representations ($x+iy$ and $re^{i\theta}$). Be comfortable with complex arithmetic, in both its cartesian and polar form. In particular, understand what the *argument* of a complex number is. Understand what the branches $arg_\tau z$ are, especially $Arg(z) = arg_{-\pi}(z)$.
(2) Know the definitions of the *modulus* and *conjugate* of a complex number. Know the geometric significance of these, and also how to use these in calculation.
(3) Know complex powers and roots. Know the geometric significance of taking the complex power or root of a complex number. Know De Moivre's Theorem (expressed in either cartesian or polar form).
(4) Know the basic definitions for planar sets: (open set, closed set, bounded set, connected set, domain). Be familiar with the use and meaning of dotted lines, open points, etc in sketching sets in the plane.
(5) Know the triangle inequality ($|z + w| \leq |z| + |w|$) and reverse triangle inequality ($|z + w| \geq | \, |z| - |w| \, |$) for complex numbers $z$ and $w$.

## 2. ANALYTIC FUNCTIONS IN GENERAL: Chapter 2

By a complex function $f(z)$ we just mean *any* function from some subset $S \subset \mathbb{C}$ to $\mathbb{C}$, which can be represented either as $f(x + iy) = u(x, y) + iv(x, y)$, or in terms of $z$, $\bar{z}$. In this chapter we gave definitions for when a complex function is *continuous* and when it is (*complex*) $differentiable$ at a given point. We observed that the basic computation rules of differentiation from calculus (e.g. product rule, quotient rule, chain rule) are obeyed. We observed that while continuity of $f$ simply meant continuity of $u$ and $v$, differentiability of $f$ was much more restrictive than $u$, $v$ being differentiable in the usual real sense.

It was then shown that, up to continuity conditions on partial derivatives, complex differentiability corresponds to $u$ and $v$ satisfying the *Cauchy Riemann equations*, which gives a convenient way to determine when/where a given function is differentiable. Finally, we discussed harmonic functions and showed that the real and imaginary parts of an analytic function were *harmonic conjugates*.

(1) Know the basic definition for $f(z)$ to be $differentiable$ at a point $z_0$, *analytic* at a point $z_0$ and *analytic* on an open set $S \subset \mathbb{C}$.

(2) Know the *Cauchy Riemann equations* and how they are used in determining differentiability and analyticity. Know how they can be used to determine restrictions on the image of an analytic function (such as: any entire function whose image is contained in a line is a constant function).
(3) Know the definition of a harmonic function, and the relation between harmonic functions and analytic functions. In particular, given a harmonic function, know how to find a harmonic conjugate.

## 3. THE ELEMENTARY ANALYTIC FUNCTIONS: Chapter 3

The most important elementary analytic functions are: *polynomials*, *rational functions*, *exponential*, *trig functions*, *logarithm* and *power functions*. Note that except for the polynomials and rational functions, the elementary analytic functions are defined using either the complex exponential $e^z$, or the complex logarithm $\log z$.

(1) Know the definitions of the elementary analytic functions, their basic properties, their domains of analyticity, and their derivatives.
(2) Know the meaning of zeros of polynomials and poles of rational functions.
(3) Understand the multi-valued-ness of $\log z$ and $z^\alpha$. In general, know what it means for a function to be a branch of a multi-valued complex function.
(4) Know what the standard branches of $\log z$ are, and their domains of analyticity. Know the corresponding branches of the power function $z^\alpha$.
(5) Reminder: for any branch $\log_\tau z$, we have $e^{\log_\tau z} = z$ for all $z$. On the other hand, $\log_\tau(e^z) = z + i2\pi k$ for some integer $k$.
(6) Reminder: for any branch $\log_\tau z$, it is not generally true that $\log_\tau(z_1 z_2) = \log_\tau z_1 + \log_\tau z_2$.
(7) Reminder: some, but not all of the usual laws of exponents are true in the complex case. For example, although we always have

$$e^{z_1} e^{z_2} = e^{z_1+z_2},$$

$(e^{z_1})^{z_2}$ does not always equal $e^{z_1 z_2}$. Similarly, although we always have

$$z^\alpha z^\beta = z^{\alpha+\beta}$$

provided the same branch is used for each power, $(z^\alpha)^\beta$ does not always equal $z^{\alpha\beta}$.
(8) Know how to determine the domain of analyticity for compositions of elementary analytic functions.

## 4. INTEGRATION OF COMPLEX FUNCTIONS: Chapter 4

Given any smooth curve $\gamma$ and continuous complex function $f(z)$ defined along $\gamma$, we define $\int_\gamma f(z)dz$ as the limit as $n \to \infty$ of the sums

$$\sum_{i=0}^{n-1} f(z_i)(z_{i+1} - z_i) \tag{0.1}$$

where the points $z_1, ..., z_n$ partition the curve $\gamma$ into segments with the partition becoming arbitrary fine as $n \to \infty$. This gives rise to the formula:

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{0.2}$$

where $z(t)$ for $a \le t \le b$ is a parametrization of $\gamma$; we require the parameterization to be admissible, meaning that i) $z(t)$ has a continuous derivative, ii) $z'(t)$ never vanishes and iii)

$z(t)$ is one-to-one. A contour $\Gamma$ is a combination of smooth curves (and/or single points), $\gamma_1, ...\gamma_n$. We write this as $\Gamma = \gamma_1 + \cdots + \gamma_n$ and define:

$$\begin{aligned} \int_\Gamma f(z)dz &= \int_{\gamma_1+\cdots+\gamma_n} f(z)dz \\ &= \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz \end{aligned} \tag{0.3}$$

A closed contour (or loop) is a contour whose initial and terminal points are the same. A simple closed contour (or simple loop) is a closed contour which intersects itself only at the initial and terminal points.

With these definitions, the basic rules of integration are true (e.g., integral of a sum is the sum of integrals, constant factors in the integrand can be taken outside of the integral).

(1) Be aware that (0.2) makes sense for any continuous $f(z)$, and that it does not depend on the parametrization of $\gamma$, provided that the parametrization is admissible.
(2) Be sure you know how to actually evaluate (0.2) given a specific $f(z)$ and $z(t)$ (technically this is done by reducing to purely real integrals. In most examples however, this will not be necessary, as the evaluation of complex integrals is simplified by the Theorems below)
(3) Know the basic integral *Estimation Theorem* (also called the *Max-length Bound*) for complex integrals (Theorem 5, p. 170).
(4) Know the basic *Fundamental Theorem for complex integrals* which says that for a continuous function in a domain, the following are equivalent: i) existence of an anti-derivative in the domain, ii) loop integrals are zero for all loops in the domain, iii) integrals on contours in the domain depend only on the endpoints and are otherwise independent of the path (Theorem 7, p. 176).

Everything mentioned above in this section so far assumes only that $f(z)$ is continuous. On the other hand, a great deal more can be said when $f(z)$ is actually analytic in a domain $D$ containing a closed contour $\Gamma$. In this case we have the following main integral Theorems:

(5) *Deformation Invariance Theorem* (when $\Gamma$ is a closed contour, we can continuously deform it to any other closed contour in $D$ without changing the integral) (Theorem 8, p. 186)
(6) *Cauchy's Integral Theorem* (when $\Gamma$ is a closed contour and $D$ is a simply connected domain, then the integral of $f$ along $\Gamma$ is zero) (Theorem 9, p. 187)
(7) *Cauchy's Integral Formulae* (the formulae represent $f$, and its derivatives, on the interior of a simple closed contour $\Gamma$ in terms of an integral along $\Gamma$) (Theorem 14, p. 204 and Theorem 19, p. 211)

These Theorems allow us to evaluate many integrals which would otherwise be difficult or impossible to compute explicitly. And we obtain results about derivatives and anti-derivatives of analytic functions:

(8) (Theorem 16, p. 209) An analytic function on a domain $D$ is in fact infinitely differentiable in $D$.
(9) (Theorem 10, p. 187) An analytic function on a simply connected domain $D$, has an antiderivative in $D$.

We also obtain a criterion for a continuous function to be analytic in terms of vanishing of loop integrals:

(10) *Morera's Theorem* If for a continuous function $f$ on a domain $D$ all loop integrals vanish, then $f$ is analytic in $D$ (Theorem 18, p. 210).

On a more fundamental level, the preceding results make profound statements on the basic nature of analytic functions, and to some extent the complex plane itself. This gives rise to

classical Theorems on the global behavior of analytic functions such as *Liouville's Theorem* and the *Maximum Modulus Principle*. And you just gotta say "Wow" when you see that the *Fundamental Theorem of Algebra* can be proven by complex analysis.

## 5. POWER SERIES AND TAYLOR SERIES: Chapter 5 (5.1-5.3)

Know the basic facts about complex series of constants, $\sum_{j=0}^{\infty} a_j$:

(1) Know the basic convergence tests ($n$-th term test, comparison test, ratio test) for complex series.
(2) Know the meaning of absolute convergence.
(3) Know the convergence properties of the following series: geometric $\sum_{j=0}^{\infty} ar^j$, harmonic $\sum_{j=1}^{\infty} \frac{1}{j}$, alternating harmonic $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j}$, inverse power $\sum_{j=1}^{\infty} \frac{1}{j^p}$

One of the most fundamental concepts in analysis is that of convergence of a sequence of functions. For a sequence of complex functions $\{f_n(z)\}$ on a set $T$, we have the following two notions of convergence. The sequence can converge *pointwise* to a function $f$ on $T$: $\lim_{n\to\infty} |f_n(z_0) - f(z_0)| = 0$ for all $z_0 \in T$. Or the sequence can converge *uniformly* to a function $f$ on $T$: $\lim_{n\to\infty} \max_{z_0 \in T} |f_n(z_0) - f(z_0)| = 0$ (compare this to the definition in the book; they are equivalent). These concepts and results are applied to the general complex *power series*.

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{0.4}$$

where the $a_j$ are complex coefficients and the series is referred to as a power series centered at $z_0$.

The natural questions are: for what $z$ does (0.4) converge as a series of complex numbers? if (0.4) does converge for all $z$ in a domain $D$, thus defining a function $f(z)$ on $D$, what properties can we expect of $f(z)$?

(1) Know the difference between uniform and pointwise convergence of a general sequence of functions $f_n$ on a domain $D$, especially applied to the sequence of partial sums $f_n(z) = \sum_{j=0}^{n} a_j (z - z_0)^j$ of a power series.
(2) Know the definition of radius of convergence $R$ or the power series (0.4); the series converges at all points in $|z - z_0| < R$ (uniformly in any smaller closed subdisc around $z_0$), and diverges when $|z - z_0| > R$. Recall that $R$ can be zero or $\infty$.
(3) Recall that the power series (0.4) always converges to an analytic function inside its radius of convergence.
(4) Know Taylor's Theorem: if $f$ is analytic in $D$, then for any $z_0 \in D$, $f$ can be represented by its Taylor series in a disk centered at $z_0$, and that the radius of convergence of this series is at least the distance from $z_0$ to the boundary of $D$. Moreover, the coefficients of this series are given by the formula:
$$a_j = \frac{f^{(j)}(z_0)}{j!}$$
(5) Note that power series representations are always unique. In other words, if two power series in a disk centered at $z_0$ converge to the same function around $z_0$, then all their coefficients must be the same. This is a result of Taylor's Theorem.
(6) Know the Taylor series expansions (and their associated radii of convergence) of the standard analytic functions, and how to use known Taylor expansions for functions $f(z)$ and $g(z)$ to determine the Taylor expansion of $(f + g)(z)$, $cf(z)$ for $c$ a constant, or $(z - z_0)^n f(z)$ for a positive integer $n$.
(7) Know that a uniform limit of analytic functions on a domain is itself analytic, and furthermore that the integral (respectively, derivative) of the limit is the limit of the integrals (respectively, derivatives).

(8) Know the following concept: an analytic function $f$ has a zero of order m at $z_0$ if the first non-zero term in (0.4) is $a_m(z-z_0)^m$, or equivalently, if $f$ can be written in a disk centered at $z_0$ as $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic and nonzero at $z_0$.

## 6. LAURENT SERIES, ISOLATED SINGULARITIES AND EVALUATION OF REAL INTEGRALS: Chapter 5 (5.5, 5.6), 6 (6.1, 6.3)

When $f(z)$ is analytic on an *annular region*

$$r < |z - z_0| < R$$

around $z_0$, then we do not have a Taylor series expansion around $z_0$ in general, since $f$ may have a singularity at $z_0$. What we have instead is a Laurent expansion of $f(z)$ in this annular region:

$$f(z) = \sum_{j=1}^{\infty} \frac{a_{-j}}{(z-z_0)^j} + \sum_{j=0}^{\infty} a_j (z-z_0)^j \tag{0.5}$$

(1) Specifically, Laurent's Theorem states: if $f(z)$ is analytic in an annular region around $z_0$ as above, then (0.5) converges to $f(z)$ in this domain (uniformly on any closed sub-annulus) where the coefficients are given by the following formula's:

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{j+1}}dz; \quad a_{-j} = \frac{1}{2\pi i}\int_C f(z)(z-z_0)^{j-1}$$

where $C$ is any positively oriented simple closed contour inside the above annulus and containing $z_0$ in its interior.

(2) Know that you can integrate and differentiate a Laurent series (in particular a Taylor series) term-by-term.

(3) Know what it means for $f$ to have an *isolated singularity* at $z_0$.

(4) Know that an isolated singularity $z_0$ is either i) removable, ii) a pole or iii) an essential singularity. Know the definitions of these in terms of the Laurent series in a punctured disk $0 < |z - z_0| < \rho$ centered at $z_0$. In particular, recall that the order of a pole $z_0$ of $f$ is $m$ where $\frac{1}{(z-z_0)^m}$ is the lowest power of $(z - z_0)$ which has a nonzero coefficient.

(5) Know the equivalent characterizations of the different types of singularities given in Theorem 18, p. 284. In particular, know what different types of behaviors are exhibited by $f$ around $z_0$ for each type of singularity.

(6) When $f$ has an isolated singularity at $z_0$, recall the definition of $Res(f, z_0)$, the Residue of $f$ at $z_0$: the coefficient of $(z - z_0)^{-1}$ in the Laurent series of $f$ on $0 < |z - z_0| < \rho$.

(7) If $f$ has a removable singularity $z_0$ or is analytic at $z_0$, then $Res(f; z_0) = 0$.

(8) If $f$ has a pole of order $m$ at $z_0$, then

$$Res(f; z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].$$

Note that this is equivalent to

$$Res(f; z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}g(z).$$

where $g(z)$ is analytic and non-zero at $z_0$ and $f(z) = \frac{g(z)}{(z-z_0)^m}$ on $0 < |z - z_0| < \rho$.

(9) Often Taylor series can be used to compute residues at isolated singularities, such as $e^{1/z}$ at $z_0 = 0$ (an essential singularity) or $(\sin z)/z^2$ at $z_0 = 0$ (a simple pole).

(10) Know that there are formulae which give $Res(f, z_0)$ when $f$ satisfies some special conditions. A special example is: when $f = g/h$, with $g, h$ analytic at $z_0$, $h$ has a zero of order 1 at $z_0$ while $g(z_0) \neq 0$, then

$$Res(f, z_0) = g(z_0)/h'(z_0).$$

(this formula follows from item (8) above)

(11) Know Cauchy's Residue Theorem: if $f$ is analytic on and inside a positively oriented simple closed contour $C$ except for finitely many singularities $z_1, .., z_n$ inside $C$, then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f, z_j)$$

Note that this just reduces to Cauchy's Integral Theorem in the case that $f$ has no singularities in $C$. And it generalizes the Cauchy Integral Formula.

(12) Evaluation of real integrals, using Cauchy's Residue Theorem:

- Improper integrals; knowing how to choose a contour, typically consisting of straight lines and parts of circles, and knowing how to evaluate or make estimates on each piece of the contour. Lemma 1 on p. 322 is useful.