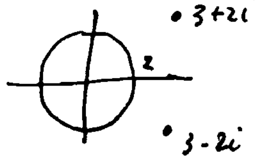
Solution to HW 7

§ 4.4.

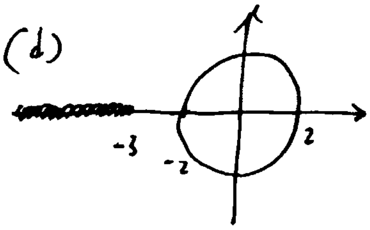
3. (a) yes (b) yes (c) no

9. (a) yes (b) no (c) yes

10. (c) $f(z)$ has singularities at the two zeros of $z^2-6z+10$, they're $3\pm 2i$ which stay outside the circle $|z|=2$.



(d)



Log$(z+3)$ is analytic on the disk $|z|\le 2$

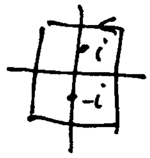
13. $$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

(a) $$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = \frac{1}{2i}\int_{\Gamma_1} \frac{dz}{z-i} - \frac{1}{2i}\int_{\Gamma_1}\frac{dz}{z+i}$$

$$= \frac{1}{2i}\,2\pi i - \frac{1}{2i}\,0 = \pi$$

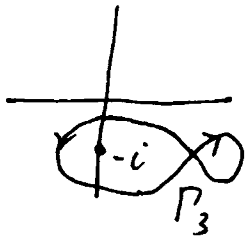
(b) $$\int_{\Gamma_2} \frac{dz}{z^2+1} = \frac{1}{2i}\int_{\Gamma_2} \frac{dz}{z-i} - \frac{1}{2i}\int_{\Gamma_2}\frac{dz}{z+i}$$

$$= \frac{1}{2i}\cdot 2\pi i - \frac{1}{2i}\,2\pi i = 0.$$



(c) $$\int_{\Gamma_3} \frac{dz}{z^2+1} = \frac{1}{2i}\int_{\Gamma_3} \frac{dz}{z-i} - \frac{1}{2i}\int_{\Gamma_3}\frac{dz}{z+i}$$

$$= \frac{1}{2i}\,0 - \frac{1}{2i}\,2\pi i = -\pi$$



(16) $$\oint_{|z|=1} f(z)\,dz = \oint_{|z|=1} \frac{A_K}{z^K}\,dz + \oint_{|z|=1} \frac{A_{K-1}}{z^{K-1}}\,dz + \cdots + \oint_{|z|=1} \frac{A_1}{z}\,dz + \oint_{|z|=1} g(z)\,dz$$

$$= \underbrace{0 + 0 + \cdots + 2\pi i\, A_1}_{\text{Example 2, Section 4.2}} + 0 \quad \leftarrow \text{Cauchy's Integral Theorem}$$

$$= 2\pi i\, A_1$$

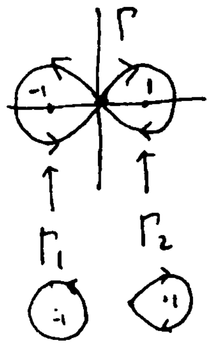
(17) $$\frac{2z^2 - z + 1}{(z-1)^2(z+1)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$\Rightarrow \quad 2z^2 - z + 1 = A(z+1) + B(z+1)(z-1) + C(z-1)^2$$

Set $z = 1 \Rightarrow 2 = A \cdot 2 \Rightarrow A = 1$

$z = 0 \Rightarrow 1 = 1 - B + C \Rightarrow B = C$

$z = -1 \Rightarrow 4 = C \cdot 4 \Rightarrow C = 1 \;\therefore B = 1$



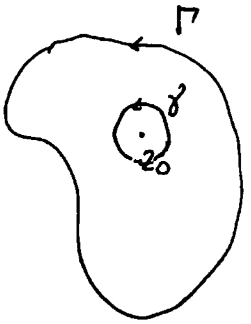
$$\int_\Gamma \frac{1}{(z-1)^2}\,dz + \frac{1}{z-1}\,dz + \frac{1}{z+1}\,dz$$

$$= \int_{\Gamma_2} \frac{1}{(z-1)^2}\,dz + \frac{1}{z-1}\,dz + \int_{\Gamma_1} \frac{1}{z+1}\,dz$$

$$= (-2\pi i) + 2\pi i = 0$$

Note: $\displaystyle\int_{\Gamma_1} \frac{1}{(z-1)^2}\,dz = \int_{\Gamma_1} \frac{dz}{z-1} = \int_{\Gamma_2} \frac{dz}{z+1} = 0.$

§ 4.5 (2) As f, g are analytic inside and on Γ. Γ is contained in some simply connected domain D. $\forall z_0$ inside Γ, there is a disk centred at z_0 which is inside of Γ as well.



Γ is deformable to γ
(the boundary of the disk)

$$\therefore\ f(z_0) \overset{\text{Cauchy}}{=} \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}\,dw$$

$\frac{f(w)}{w-z_0}$ is analytic away from z_0

$$= \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z_0}\,dw$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{g(w)}{w-z_0}\,dw$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z_0}\,dw$$

$$\overset{\text{Cauchy Integral form}}{=} g(z_0)$$

$\because$ z_0 is arbitrary

$\therefore$ $g(z) = f(z)$ inside Γ.

③ (d) $$\oint_C \frac{5z^2+2z+1}{(z-i)^3}\,dz = \frac{2\pi i}{2!}\left(5z^2+2z+1\right)''\Big|_{z=i}$$

$$= 10\pi i.$$

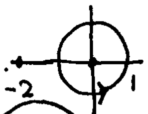
(f) $$\oint_C \frac{\sin z}{z^2(z-4)}\,dz = \int_C \frac{\left(\frac{\sin z}{z-4}\right)}{z^2}\,dz$$

$$= 2\pi i\left(\frac{\sin z}{z-4}\right)'\Big|_{z=0}$$

$$= 2\pi i\left(\frac{\cos z}{z-4}\Big|_{z=0} - \frac{\sin z}{(z-4)^2}\Big|_{z=0}\right)$$

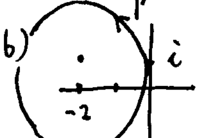
$$= -\frac{\pi i}{2}$$

④ (a)



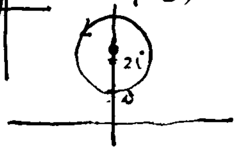
$f(z) = \frac{z+i}{z+2}$, $\quad \int_C \frac{\left(\frac{z+i}{z+2}\right)}{z^2}\,dz = 2\pi i\, f'(0) = \frac{\pi}{2} + \pi i$

(b)



$f(z) = \frac{z+i}{z^2}$, $\quad \int_C \frac{\left(\frac{z+i}{z^2}\right)}{z+2}\,dz = 2\pi i\, f(-2) = -\frac{\pi}{2} - \pi i$

(c)



$\int_C \frac{z+i}{z^3+2z^2}\,dz = 0.$

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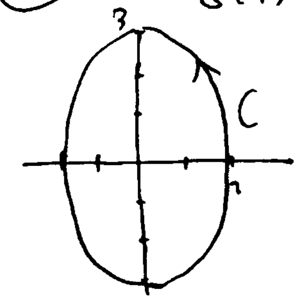

$$G(1) = \int_C \frac{\zeta^2-\zeta+2}{\zeta-1}\,d\zeta = 2\pi i\,(1^2-1+2) = 4\pi i$$

$$G'(i) = \oint_C \frac{\zeta^2-\zeta+2}{(\zeta-i)^2}\,d\zeta$$

$$= 2\pi i\,(\zeta^2-\zeta+2)'\Big|_{\zeta=i}$$

$$= 2\pi i\,(2i-1) = -4\pi - 2\pi i$$

$$G''(i) = 2\oint_C \frac{\zeta^2-\zeta+2}{(\zeta-i)^3}\,d\zeta = 2\cdot\frac{2\pi i}{2!}\,(\zeta^2-\zeta+2)''\Big|_{\zeta=i}$$

$$= 4\pi i$$

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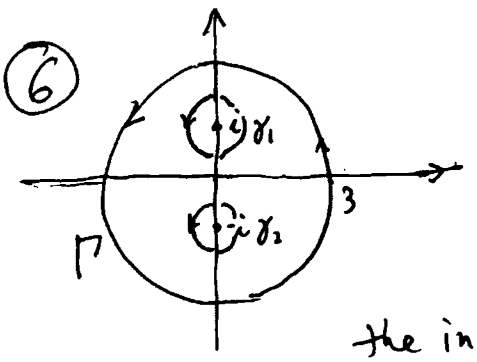
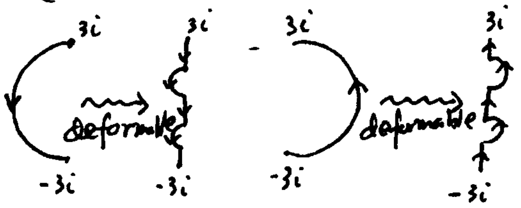


$$(z^2+1) = (z+i)(z-i)$$



the integrand is analytic in

 and 

$$\therefore \oint_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz = \int_{\gamma_1} \frac{e^{iz}/(z+i)^2}{(z-i)^2}\,dz + \int_{\gamma_2} \frac{e^{iz}/(z-i)^2}{(z+i)^2}\,dz$$

$$= 2\pi i \left(\frac{e^{iz}}{(z+i)^2}\right)'_{z=i} + 2\pi i \left(\frac{e^{iz}}{(z-i)^2}\right)'_{z=-i}$$

$$= 2\pi i \left[\frac{i e^{iz}}{(z+i)^2}\Big|_{z=i} - 2\,\frac{e^{iz}}{(z+i)^3}\Big|_{z=i}\right] + 2\pi i \left[\frac{i e^{iz}}{(z-i)^2}\Big|_{z=-i} - 2\,\frac{e^{iz}}{(z-i)^3}\Big|_{z=-i}\right] = \frac{\pi}{e}.$$