Solution to HW Set 3

Section 2.1

(1) (i) $\dfrac{1}{z^2} = \dfrac{1}{(x+iy)^2} = \dfrac{(x-iy)^2}{(x^2+y^2)^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2} + i\,\dfrac{-2xy}{(x^2+y^2)^2}$

domain of definition: $\mathbb{C}\setminus\{0\}$

(ii) $e^{2z+3i} = e^{2x+(2y+3)i}$

$= e^{2x}\cos(2y+3) + i\,e^{2x}\sin(2y+3)$

domain of definition: $\mathbb{C}$

#6. p.57 (a) $J\left(\frac{1}{z}\right) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{\frac{1}{z}}\right) = \frac{1}{2}\left(\frac{1}{z} + z\right) = J(z)$

(b) Write $z = e^{i\theta}$ as $|z| = 1$. Then

$J(e^{i\theta}) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta$

For $\theta \in [0, 2\pi]$, the range of $\cos\theta$ is $[-1, 1]$.

(c) For $z = re^{i\theta}$

$J(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right)$

$= \frac{1}{2}\left(\left(r\cos\theta + \frac{1}{r}\cos(-\theta)\right) + i\left(r\sin\theta + \frac{1}{r}\sin(-\theta)\right)\right)$

$= \underbrace{\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta}_{u} + i\cdot\underbrace{\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta}_{v}$

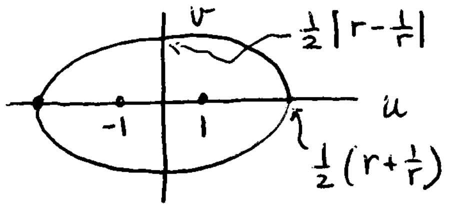
$\therefore$ If $r>0, r\neq 1$,

We have: $\dfrac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} + \dfrac{v^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2} = \cos^2\theta + \sin^2\theta = 1$

This is the equation for an ellipse in the uv-plane

$\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2 - \left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2 = \frac{1}{4}\left(r^2+2+\frac{1}{r^2}\right) - \frac{1}{4}\left(r^2-2+\frac{1}{r^2}\right) = 1$

$\therefore$ the ellipse has foci at ± 1.



Section 2.2

(1) (a) $\lim\limits_{z\to 2i} \dfrac{z^2+4}{z-2i} = \lim\limits_{z\to 2i} \dfrac{(z+2i)(z-2i)}{z-2i}$

$= \lim\limits_{z\to 2i} (z+2i) = 4i$

(b) $\lim\limits_{z\to i} \left|\dfrac{z^3-i}{z^6+1}\right| = \lim\limits_{z\to i} \left|\dfrac{z^3-i}{(z^3-i)(z^3+i)}\right|$

$= \lim\limits_{z\to i} \dfrac{1}{|z^3+i|} = \infty$

as $\lim\limits_{z\to i} |z^3+i| = 0$

(c) $\lim\limits_{\Delta z\to 0} \dfrac{(z_0+\Delta z)^3 - z_0^3}{\Delta z} = \lim\limits_{\Delta z\to 0} \dfrac{\left[z_0^3+3z_0^2\Delta z+3z_0(\Delta z)^2+(\Delta z)^3\right]-z_0^3}{\Delta z}$

$= \lim\limits_{\Delta z\to 0} \left[3z_0^2 + 3z_0\cdot\Delta z + (\Delta z)^2\right] = 3z_0^2$

(d) $\lim\limits_{z\to 2\pi} (e^{iz} - e^{-iz}) = e^{2\pi i} - e^{-2\pi i}$

$= (\cos 2\pi + i\sin 2\pi) - (\cos(-2\pi) + i\sin(-2\pi))$

$= (1+0) - (1+0) = 0$

Section 2.3

\#2 If $\lambda(z) = \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$, then as $z\to z_0$

we have $\lambda(z)\to 0$ by the definition of $f'(z_0)$.

We also have $f(z) = f(z_0) + f'(z_0)(z-z_0) + \lambda(z)(z-z_0)$

by multiplying $z-z_0$ and rearranging terms.

#4. (b)

$$\frac{\operatorname{Im}(z_0+\Delta z) - \operatorname{Im} z_0}{\Delta z} = \frac{\operatorname{Im}\Delta z}{\Delta z}$$

$$= \frac{\Delta y}{\Delta x + i\Delta y}$$

Take $\Delta y = k\Delta x$ for some constant k

so $\Delta y \to 0$ as $\Delta x \to 0$. Observe

$$\frac{\Delta y}{\Delta x + i\Delta y} = \frac{k\Delta x}{\Delta x + ik\Delta x} = \frac{k}{1+ik}$$

Then take $k=0$, we get $\frac{k}{1+ik} = 0$

$k = 1$ we get $\frac{k}{1+ik} = \frac{1}{1+i} \neq 0$

$\therefore$ $\frac{\operatorname{Im}(z_0+\Delta z) - \operatorname{Im} z_0}{\Delta z}$ does not have a limit as $\Delta z \to 0$ for any $z_0 \in \mathbb{C}$

$\therefore$ $\operatorname{Im} z$ is nowhere differentiable.

(c)

$$\frac{|z_0+\Delta z| - |z_0|}{\Delta z} = \frac{\sqrt{(x_0+\Delta x)^2 + (y_0+\Delta y)^2} - \sqrt{x_0^2+y_0^2}}{\Delta x + i\Delta y}$$

$$= \frac{(x_0+\Delta x)^2 + (y_0+\Delta y)^2 - (x_0^2+y_0^2)}{(\Delta x + i\Delta y)\left(\sqrt{(x_0+\Delta x)^2+(y_0+\Delta y)^2} + \sqrt{x_0^2+y_0^2}\right)}$$

$$= \frac{2x_0\Delta x + (\Delta x)^2 + 2y_0\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)\left(\sqrt{(x_0+\Delta x)^2+(y_0+\Delta y)^2} + \sqrt{x_0^2+y_0^2}\right)} \qquad (*)$$

Case 1: $z_0 = 0$. Then from $(*)$,

$$\frac{|z_0+\Delta z| - |z_0|}{\Delta z} = \frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x + i\Delta y)\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)} = \begin{cases} 1, & \text{if } \Delta y = 0,\ \Delta x > 0 \\ \frac{1}{i}, & \text{if } \Delta x = 0,\ \Delta y > 0 \end{cases}$$

$\therefore$ $|z|$ is not differentiable at 0

Case 2 $z_0 \neq 0$. Then from $(*)$

$$\frac{|z_0+\Delta z| - |z_0|}{\Delta z} \text{ tends to } \begin{cases} \frac{x_0}{\sqrt{x_0^2+y_0^2}}, & \text{if } \Delta y = 0,\ \Delta x \to 0 \\ \frac{y_0}{i\sqrt{x_0^2+y_0^2}}, & \text{if } \Delta x = 0,\ \Delta y \to 0 \end{cases}$$

Finally, $\dfrac{x_0}{\sqrt{x_0^2+y_0^2}} \neq \dfrac{y_0}{i\sqrt{x_0^2+y_0^2}}$ unless $x_0 = y_0 = 0$

$\uparrow$ real $\qquad \uparrow$ purely imaginary

But $x_0 = y_0 = 0$ cannot happen in Case 2

$\therefore$ $|z|$ is not differentiable at any $z_0 \neq 0$

$\therefore$ Case 1 and Case 2 $\Rightarrow$ $|z|$ is Nowhere differentiable

#7. (b) $\left[(z^2-3i)^{-6}\right]' = -6\cdot(z^2-3i)^{-6-1}\cdot(2z)$

$= -12z(z^2-3i)^{-7}$

(e) $\left[6i(z^3-1)^4(z^2+iz)^{100}\right]' = 6i\cdot 4(z^3-1)^3\cdot 3z^2(z^2+iz)^{100}$

$+6i(z^3-1)^4\cdot 100(z^2+iz)^{99}\cdot(2z+i)$

$= 24i(z^3-1)^3(z^2+iz)^{99}\left(3z^2(z^2+iz)+25(z^3-1)(2z+i)\right)$

$= 24i(z^3-1)^3(z^2+iz)^{99}(53z^4+28iz^3-50z-25i)$

#14 $\displaystyle\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}$ (because $f(z_0)=0$, $g(z_0)=0$)

$\displaystyle = \lim_{z\to z_0}\frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}}$

$\displaystyle = \frac{f'(z_0)}{g'(z_0)}$ (because $f'(z_0)$, $g'(z_0)$ exist and $g'(z_0)\neq 0$)