Section 5.2:

7. For any z with $|z| < 1$, both $1+z$ and $1-z$ must be in the disk $\{|z-1| < 1\}$, and so have strictly positive real part. This means that $\mathrm{Arg}(1+z)$ and $\mathrm{Arg}(1-z)$ are in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

From earlier in the book, we know that $\mathrm{Arg}(1+z) - \mathrm{Arg}(1-z) = \arg(\frac{1+z}{1-z})$, in the sense that it is *some* value of the argument function of $\frac{1+z}{1-z}$. The problem is that it might not be the principal value. To check that it is the principal value, we need to know that $-\pi < \mathrm{Arg}(1+z) - \mathrm{Arg}(1-z) \leq \pi$. However, since we showed above that $\mathrm{Arg}(1+z)$ and $\mathrm{Arg}(1-z)$ are in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for any z with $|z| < 1$, we know that in fact $-\pi < \mathrm{Arg}(1+z) - \mathrm{Arg}(1-z) < \pi$, and so $\mathrm{Arg}(1+z) - \mathrm{Arg}(1-z) = \mathrm{Arg}(\frac{1+z}{1-z})$ for any z with $|z| < 1$.

Then, for any z with $|z| < 1$,

$$\begin{aligned}\mathrm{Log}\left(\frac{1+z}{1-z}\right) &= \mathrm{Log}\left|\frac{1+z}{1-z}\right| + i\mathrm{Arg}\left(\frac{1+z}{1-z}\right)\\ &= \mathrm{Log}\frac{|1+z|}{|1-z|} + i(\mathrm{Arg}(1+z) - \mathrm{Arg}(1-z))\end{aligned}$$

$$= \mathrm{Log}|1+z| - \mathrm{Log}|1-z| + i\mathrm{Arg}(1+z) - i\mathrm{Arg}(1-z) = \mathrm{Log}(1+z) - \mathrm{Log}(1-z)$$

It is fairly straightforward to find the Taylor series expansions of $g(z) = \mathrm{Log}(1+z)$ and $h(z) = \mathrm{Log}(1-z)$ by taking nth derivatives, and so we omit the work here. Since $g(z)$ is nonanalytic on the ray $\{z = x+yi \ : \ x \leq -1, \ y = 0\}$, its Taylor series around $z_0 = 0$

$$g(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

converges in $\{|z| < 1\}$. Since $h(z)$ is nonanalytic on the ray $\{z = x+yi \ : \ x \geq 1, \ y = 0\}$, its Taylor series around $z_0 = 0$

$$h(z) = \sum_{n=1}^{\infty} \frac{-1}{n} z^n$$

also converges in $\{|z| < 1\}$. Since $\mathrm{Log}(\frac{1+z}{1-z}) = \mathrm{Log}(1+z) - \mathrm{Log}(1-z)$ for any z in this disk, we know that the series

$$\begin{aligned}\mathrm{Log}(1+z) - \mathrm{Log}(1-z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n - \sum_{n=1}^{\infty} \frac{-1}{n} z^n\\ &= \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right) - \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots\right)\\ &= 2z + \frac{2z^3}{3} + \frac{2z^5}{5} + \ldots = \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{2n+1}\end{aligned}$$

or: By Theorem 5 part (ii),

converges to $\mathrm{Log}(\frac{1+z}{1-z})$ inside this disk as well. By Theorem 11, this power series is then the Taylor series for $\mathrm{Log}(\frac{1+z}{1-z})$, and the radius of convergence R is at least 1. Notice that $\mathrm{Log}(\frac{1+z}{1-z})$ is nonanalytic at $z = 1$, and so the radius of convergence R cannot possibly be more than 1. Therefore, $R = 1$.

Section 5.3:

3(a). $\lim_{j\to\infty}|\frac{a_{j+1}}{a_j}| = \lim_{j\to\infty}|\frac{(j+1)^3}{j^3}| = \lim_{j\to\infty}\frac{(j+1)^3}{j^3} = 1$. Therefore, by problem 2, the radius of convergence is $R = \frac{1}{L} = 1$. Since the power series is centered at 0, the circle of convergence is $\{|z| = 1\}$.

3(c). $\lim_{j\to\infty}|\frac{a_{j+1}}{a_j}| = \lim_{j\to\infty}|\frac{(j+1)!}{j!}| = \lim_{j\to\infty} j + 1 = \infty$. Therefore, by problem 2, the radius of convergence is $R = \frac{1}{L} = 0$. (Though it is not technically correct to say that $\frac{1}{\infty} = 0$, it still gives the correct radius of convergence.) Since the power series is centered at 0, the circle of convergence is $\{|z| = 0\}$, meaning that this power series converges only at $z = 0$.

3(e). $\lim_{j\to\infty}|\frac{a_{j+1}}{a_j}| = \lim_{j\to\infty}\left|\frac{(3-i)\,j^2}{(j+1)^2}\right| = |\;3-i\;| = \sqrt{10}$. Therefore, by problem 2, the radius of convergence is $R = \frac{1}{L} = \frac{1}{\sqrt{10}}$ Since the power series is centered at -2, the circle of convergence is $\{|z+2| = \frac{1}{\sqrt{10}}\}$.

5(a). It is not hard to check, using the same methods as in problem 3, that the disk of convergence for this power series is $\{|z| < 3\}$. Since the power series converges to $f(z)$ inside this disk, it is automatically the Taylor series for $f(z)$. Therefore,

$$\sum_{k=0}^{\infty}\frac{k^3}{3^k}z^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k.$$

By matching up the coefficients of z^6 in these power series, we see that $\frac{6^3}{3^6} = \frac{f^{(6)}(0)}{6!}$, so $f^{(6)}(0) = \frac{6^3}{3^6}\cdot 6! = \frac{216}{729}\cdot 720 = \frac{640}{3}$.

5(b). Since the contour $\{|z| = 1\}$ lies entirely within the disk of convergence for the power series defining $f(z)$, it is ok to replace $f(z)$ by its power series representation. Then,

$$\int_{|z|=1}\frac{f(z)}{z^4}\,dz = \int_{|z|=1} z^{-4}\Big[\sum_{k=0}^{\infty}\frac{k^3}{3^k}z^k\Big]\,dz = \int_{|z|=1}\Big[\sum_{k=0}^{\infty}\frac{k^3}{3^k}z^{k-4}\Big]\,dz.$$

Since the convergence of the power series on the contour is uniform, we can switch the order of the sum and integral:

$$\int_{|z|=1}\left[\sum_{k=0}^{\infty}\frac{k^3}{3^k}z^{k-4}\right]dz=\sum_{k=0}^{\infty}\int_{|z|=1}\frac{k^3}{3^k}z^{k-4}\,dz=\sum_{k=0}^{\infty}\frac{k^3}{3^k}\int_{|z|=1}z^{k-4}\,dz.$$

But the only power of z which yields a non-zero result when integrated over a positively oriented circle centered at the origin is z^{-1}, which has an integral of $2\pi i$. Therefore, the only value of k which contributes a nonzero term to the sum is $k = 3$. Then,

$$\sum_{k=0}^{\infty}\frac{k^3}{3^k}\int_{|z|=1}z^{k-4}\,dz=\frac{3^3}{3^3}\int_{|z|=1}z^{-1}\,dz=\frac{3^3}{3^3}2\pi i=2\pi i.$$

8. Since f is analytic at the origin, there is a disk $\{|z| < R\}$ in which f is analytic. Therefore, the Taylor expansion

$$f(z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\ldots$$

converges in $\{|z| < R\}$. Since $f(0) = f'(0) = 0$, the first two terms of this Taylor expansion are 0. This means that

$$f(z)=z^2\left[\frac{f''(0)}{2!}+\frac{f'''(0)}{3!}z+\ldots\right]$$

converges in $\{|z| < R\}$, and so the power series inside the brackets converges in this disk as well. Call the power series inside the brackets $g(z)$. Then $g(z)$ converges inside this disk, and is automatically analytic within this disk. In particular, we have shown that $f(z) = z^2 g(z)$, where $g(z)$ is analytic at $z = 0$.

4. No, $|(2+3i)-0| = \sqrt{13}$ $\quad \sqrt{13} > \sqrt{10}$

$|(3-i)-0| = \sqrt{10}$.

The conclusion then follows from Lemma 2, p. 253.

2. $$\lim_{j\to\infty}\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right|=\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right|\cdot|z-z_0|$$

$$= L\cdot|z-z_0|$$

The ratio test implies: the series is convergent if $L\cdot|z-z_0|<1$ and is divergent if $L\cdot|z-z_0|>1$.

∴ the radius of convergence is $\frac{1}{L}$.