Math 300: Homework #8, SOLUTIONS

Section 4.6:

#6:

If f is entire, then so is $f^{(5)}$ (by Theorem 16). If $f^{(5)}$ is also bounded, then by Liouville's Theorem, $f^{(5)}$ is constant. But then $f^{(4)}$ is an anti-derivative of a constant and thus must be a polynomial of degree 1. Then $f^{(3)}$ must be a polynommial of degree 2. Continuing we see that f must be a polynomial of degree 5.

#8: $f(z)/(3z^2)$ is analytic on the closed annulus $1 \leq |z| \leq 2$. The boundary consists of two circles and $|f(z)/(3z^2)| \leq 1$ on both of these circles. Thus, by the maximum modulus principle (specifically, Theorem 24), $|f(z)/(3z^2)| \leq 1$ on the entire closed annulus.

#10: f is analytic on $D = \{|z| < R\}$ and its absolute value is upper bounded by 1 on D. If $f(0) = i$, then f attains its maximum in D. By the maximum modulus principle (specifically Theorem 23), f is constant in D. So, the only such function is the constant function $f(z) = i$ for all $z \in D$.

Section 5.1:

#2a:

The ratio $c_{j+1}/c_j = 1/(j+1) \to 0 < 1$. Thus, the series converges by the ratio test.

#2b: The ratio $|c_{k+1}/c_k| = |3+i|/(k+1) \to 0 < 1$. Thus, the series converges by the ratio test.

#11c: The ratio $|c_{j+1}/c_j| = |z|/(j+1) \to 0 < 1$. Thus, the series converges for all z.

#11d: The ratio $|c_{k+1}/c_k| = |z+5i|^2((k+2)/(k+1))^2 \to |z+5i|^2$. Thus, the series converges for all z such that $|z+5i|^2 < 1$, which is the circle of radius 1 centered at $-5i$.

#13: Let $f_N(x)$ be the function which takes value $1/j^p$ on the interval $[j-1, j)$ for $j = 2, \ldots N$. Then $f_N(x) \leq (1/x^p)$ on the interval $[1, N)$. Thus,

$$\sum_{j=2}^{N}(1/j^p) \leq \int_1^N (1/x^p)dx \leq \int_1^\infty (1/x^p)dx = \frac{1}{x^{p-1}(-p+1)}\Big]_1^\infty = \frac{1}{p-1}$$

since $p > 1$. Thus, $\sum_{j=2}^\infty (1/j^p)$ converges and it follows that so does $\sum_{j=1}^\infty (1/j^p)$.

Section 5.2:

#1a and #2a:

$f(z) = e^{-z}$. By induction, one shows that $f^{(j)}(z) = (-1)^j e^{-z}$, and so $f^{(j)}(0) = (-1)^j$. Thus, the Taylor series for $f(z)$ around $z_0 = 0$ is $\sum_{j=0}^\infty (-1)^j z^j/(j!)$. Since $f(z)$ is analytic on $\mathbb{C}$, by Theorem 3, the Taylor series is valid on all of $\mathbb{C}$.

#1d and #2d:

$f(z) = 1/(1-z)$. By induction, one shows that $f^{(j)}(z) = j!/(1-z)^{j+1}$, and so $f^{(j)}(i) = j!/(1-i)^{j+1}$, Thus, the Taylor series for $f(z)$ around $z_0 = i$ is $\sum_{j=0}^\infty (z-i)^j/(1-i)^{j+1}$. Since $f(z)$ is analytic on $D = \{|z-1| < 1\}$, by Theorem 3, the Taylor series is valid on D.

#1e and #2e:

$f(z) = Log(1-z)$. By induction, one shows that for $j \geq 1$, $f^{(j)}(z) = -(j-1)!/(1-z)^j$, and so $f^{(j)}(0) = -(j-1)!$ Thus, the Taylor series for $f(z)$ around $z_0 = 0$ is $\sum_{j=0}^{\infty} -z^j/j$. Since $f(z)$ is analytic on $D = \{|z-1| < 1\}$, by Theorem 3, the Taylor series is valid on D.

#8a

$$\sin(-z) = \sum_{j=0}^{\infty}(-1)^j(-z)^{2j+1}/((2j+1)!) = \sum_{j=0}^{\infty}(-1)^{3j+1}z^{2j+1}/((2j+1)!) = \sum_{j=0}^{\infty}(-1)^{j+1}z^{2j+1}/((2j+1)!)$$

$$= -\sum_{j=0}^{\infty}(-1)^j z^{2j+1}/((2j+1)!) = -\sin(z)$$

#8b: By Theorem 4, we can termwise differentiate the Taylor series for e^z. Since $e^z = \sum_{j=0}^{\infty} z^j/(j!)$, we have

$$\frac{de^z}{dz} = \sum_{j=0}^{\infty} jz^{j-1}/(j!) = \sum_{j=1}^{\infty} z^{j-1}/((j-1)!) = \sum_{j=0}^{\infty} z^j/(j!) = e^z$$