Math 300: Homework #6: Solutions

1. Section 4.1 #8

   Solution: A parameterization for the contour in Figure 4.14 is:

   $$z(t) = \left\{ \begin{array}{cc} -2+2i+t(3-2i) & 0 \le t \le 1 \\ -e^{-i(t-1)\pi} & 1 \le t \le 2 \end{array} \right\}$$

   A parameterization for the opposite contour is:

   $$z(t) = \left\{ \begin{array}{cc} -2+2i-t(3-2i) & -1 \le t \le 0 \\ -e^{i(t+1)\pi} & -2 \le t \le -1 \end{array} \right\}$$

2. Let $\gamma = \{(x,y) : y = x^{1/3} : \ -1 \le x \le 1\}$.

   (a) Sketch γ.

   Solution: To be provided in separate file.

   (b) Verify that γ is a smooth arc, by showing that $z(t) = t^3 + it$, $-1 \le t \le 1$ is an admissible parameterization of γ

   Solution:

   i): $z'(t) = 3t^2 + i$, which is continuous since the real and imaginary parts are polynomials.

   ii) $z'(t)$ is never zero since its imaginary part is always 1.

   iii) If $z(t_1) = z(t_2)$, then the imaginary parts of $z(t_1)$ and $z(t_2)$ must be equal. Thus, $t_1 = t_2$, and so $z(t)$ is 1-1.

3. Let $\gamma = \{(x,y) : y = x^{2/3} : \ -1 \le x \le 1\}$.

   (a) Sketch γ.

   Solution: To be provided in separate file.

   (b) Show that $z(t) = t^3 + it^2$, $-1 \le t \le 1$ is a parameterization of γ that satisfies admissability properties i) and iii), but not ii).

   i): $z'(t) = 3t^2 + 2ti$, which is continuous since the real and imaginary parts are polynomials.

   ii) $z'(0) = 0$ (so, property (ii) does not hold).

   iii) If $z(t_1) = z(t_2)$, then the real parts of $z(t_1)$ and $z(t_2)$ must be equal. Thus, $t_1^3 = t_2^3$ and so $t_1 = t_2$. Thus, $z(t)$ is 1-1.

(c) Show that γ is *not* a smooth arc. Hint: Suppose that $z(t) = u(t) + iv(t)$ is a parameterization of γ such that $z'(t)$ exists and is continuous. Show that for some t_0, $u(t_0) = 0 = v(t_0)$ and for this t_0, $u'(t_0) = 0 = v'(t_0)$.

Solution: Suppose that $z(t) = u(t) + iv(t)$, $\{z(t) : a \leq t \leq b\} = \{(x, y) : y = x^{2/3} : \ -1 \leq x \leq 1\}$, $z'(t)$ exists and is continuous and $z(t)$ is 1-1. Then, for some $a \leq t_0 \leq b$, $z(t_0) = 0$. We claim that $z'(t_0) = 0$.

We have $v(t) = u(t)^{2/3}$. Thus, $v'(t) = (2/3)\frac{u'(t)}{u(t)^{1/3}}$. Since $u(t_0) = 0$, we have $u'(t_0) = 0$, for otherwise $v'(t_0) = \infty$, contrary to the assumption that $z'(t)$ exists and is continuous.

Since $z(t)$ is 1-1 and continuous and $v = u^{2/3}$, $u(t)$ is 1-1 and continuous. Thus, it is either always increasing or always decreasing. Without loss of generality, we may assume that $u(t)$ is always increasing. Thus, for $t \leq t_0$, $v(t)$ is decreasing and so the derivative of $v(t)$ at t_0 from the left is non-positive. Similarly, for $t \geq t_0$, $v(t)$ is increasing and so the derivative of $v(t)$ at t_0 from the right is non-negative. Thus, since $v'(t_0)$ exists, it must be 0.

So, $u'(t_0) = 0$ and $u'(t_0) = 0$, and so $z'(t_0) = 0$.

4. Section 4.2 #6

Solution:

(a): parameterize the counterclockwise circle Γ of radius 2, centered at 0 as: $z(t) = 2e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\int_\Gamma \overline{z} dz = \int_0^{2\pi} 2e^{-it}(2ie^{it})dt = 4i \int_0^{2\pi} dt = 8\pi i.$$

(b): for the clockwise circle Λ of radius 2, centered at 0, we have $\Lambda = -\Gamma$. Thus, $\int_\Lambda \overline{z} dz = -\int_\Gamma \overline{z} dz = -8\pi i$.

(c): for the 3-times clockwise circle Ω of radius 2, we have $\Omega = \Lambda + \Lambda + \Lambda$. Thus, $\int_\Omega \overline{z} dz = 3 \int_\Lambda \overline{z} dz = -24\pi i$.

5. Section 4.2 #8

Solution: $C = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where the γ_i are parameterized as follows.

$\gamma_1 : \quad z(t) = t, 0 \leq t \leq 1$

$\gamma_2 : \quad z(t) = 1 + ti, 0 \leq t \leq 1$

$\gamma_3 : \quad z(t) = 1 + i - t, 0 \leq t \leq 1$

$\gamma_4 : \quad z(t) = i - ti, 0 \leq t \leq 1$

$\int_{\gamma_1} e^z dz = \int_0^1 e^t e^t dt = (1/2)e^{2t} \,]_0^1 = (1/2)e^2 - 1/2.$

$\int_{\gamma_2} e^z dz = \int_0^1 e^{1+ti} e^{1+ti} i dt = (1/2)e^{2+2ti} \,]_0^1 = (1/2)e^{2+2i} - (1/2)e^2.$

$\int_{\gamma_3} e^z dz = \int_0^1 e^{1+i-t} e^{1+i-t}(-1) dt = (1/2)e^{2+2i-2t} \,]_0^1 = (1/2)e^{2i} - (1/2)e^{2+2i}.$

$\int_{\gamma_4} e^z dz = \int_0^1 e^{i-ti} e^{i-ti}(-i) dt = (1/2)e^{2i-2ti} \,]_0^1 = (1/2) - (1/2)e^{2i}.$

So, $\int_{\gamma_4} e^z = 0$.

Note: This could also be deduced from Theorem 7 (since e^z has an anti-derivative, namely itself).

6. Let $f(z) = \frac{z+1}{z}$. For each of the following directed smooth curves, find an admissible parameterization and compute $\int_\gamma f(z)dz$.

   (a) The semi-circle in the upper half-plane, centered at 0, with radius 1, in the clockwise direction.

   Solution: re-write $f(z) = 1 + 1/z$.

   $z(t) = -e^{-it}, \ 0 \leq t \leq \pi$

   $\int_\gamma f(z) = \int_0^\pi (1 - e^{it})(i)e^{-it} dt = -e^{-it} - it \,]_0^\pi = 1 - \pi i + 1 = 2 - \pi i$

   (b) The semi-circle in the lower half-plane, centered at 0, with radius 1, in the clockwise direction.

   $z(t) = e^{-it}, \ 0 \leq t \leq \pi$

   $\int_\gamma f(z) = \int_0^\pi (1+e^{it})(-i)e^{-it} dt = e^{-it} - it \,]_0^\pi = -1 - \pi i - 1 = -2 - \pi i$

   (c) The circle, centered at 0, with radius 1, in the clockwise direction.

   $z(t) = e^{-it}, \ 0 \leq t \leq 2\pi$

   $\int_\gamma f(z) = \int_0^{2\pi} (1+e^{it})(-i)e^{-it} dt = e^{-it} - it \,]_0^{2\pi} = 1 - 2\pi i - 1 = -2\pi i$

   (d) The circle, centered at 0, with radius 1, in the counter-clockwise direction.

   $z(t) = e^{it}, \ 0 \leq t \leq 2\pi$

   $\int_\gamma f(z) = \int_0^{2\pi} (1 + e^{-it})(i)e^{it} dt = e^{it} + it \,]_0^{2\pi} = 1 + 2\pi i - 1 = 2\pi i$

7. Section 4.2 #14

   (a) $\ell(C) = 6\pi$.

   $|z^2| = |(z^2 - i) + i| \leq |z^2 - i| + 1$.

   Thus, for z on the circle of radius 3, centered at 0,

   $|z^2 - i| \geq |z^2| - 1 = 9 - 1 = 8$.

   So, the absolute value of the integrand is upper-bounded by 1/8. Thus, the absolute value of the integral is upper bounded by $(1/8)\ell(C) = 3\pi/4$.

   (b) The length of the line segment from R to $R + 2\pi i$ is 2π.

   For all z on this line segment, $|e^z| = e^R$ and $|e^{3z}| = e^{3R}$.

   And $|e^z| = |(e^z - 1) + 1| \leq |e^z - 1| + 1$. So,

   $|e^z - 1| \geq e^R - 1$. Thus,

   $$|\frac{e^{3z}}{1+e^z}| \leq \frac{e^{3R}}{e^R - 1}$$

   Thus,

   $$|\int_\gamma \frac{e^{3z}}{1+e^z} dz| \leq (\frac{e^{3R}}{e^R - 1})(2\pi)$$

   (c) The length of the arc Γ of the circle $|z| = 1$ in the first quadarant is $\pi/2$.

   For $z \in \Gamma$, $Log(z) = iArg(z)$, $0 \leq Arg(z) \leq \pi/2$. Thus, $|Log(z)| \leq \pi/2$.

   Thus, $|\int_\Gamma Log(z)dz| \leq (\pi/2)(\pi/2) = \pi^2/4$

   (d) The length of the line segment γ from 0 to i is 1. From the definition of $\sin(z)$, one sees that for $z \in \gamma$, $Re(\sin(z)) = 0$, and so $|e^{\sin(z)}| = 1$. Thus, $|\int_\gamma e^{\sin(z)} dz| \leq 1 \cdot 1 = 1$.

8. Section 4.3 #2

   Solution: If $P(z)$ is a polynomial, then it has an anti-derivative, and thus by Theorem 7 all loop integrals are 0.

9. Section 4.3 #4

   Solution: False. Let $f(z) = 1/z$ and Γ be the counter-clockwise circle of radius 1 centered at 0. Then $\int_\Gamma f(z)dz = 2\pi i \neq 0$.

10. Section 4.3 #11

    Solution: $(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$ and so the right-hand side is a continuous function which has an anti-derivative, namely the left-hand side. Thus, by Theorem 6,

    $$\int_\Gamma (f'(z)g(z) + f(z)g'(z))dz = f(z)g(z)\ \Big]_{z_I}^{z_T}$$

    which is equivalent to the integration by parts formula.