Solution to Set 5

§ 3.2.

5 (c) $\sin(2i) = \dfrac{e^{i(2i)} - e^{-i(2i)}}{2i} = \dfrac{e^{-2} - e^{2}}{2i} = \dfrac{e^{2} - e^{-2}}{2} i$

(d) $\cos(1-i) = \dfrac{e^{i(1-i)} + e^{-i(1-i)}}{2} = \dfrac{e^{1+i} + e^{-1-i}}{2}$

$= \dfrac{e(\cos(+1) + i\sin(+1)) + \frac{1}{e}(\cos(-1) + i\sin(-1))}{2}$

$= \cos 1 \cdot \dfrac{e + \frac{1}{e}}{2} + i \sin 1 \cdot \dfrac{e - \frac{1}{e}}{2}$

9 (a) $w' = 2\pi z\, e^{\pi z^2}$

(b) $w' = -2\sin(2z) - i \cdot \frac{1}{z^2} \cos\frac{1}{z}$

(c) $w' = 2 \cdot \cos 2z \cdot e^{\sin(2z)}$

17. (b) $e^{iz} = 3 \Rightarrow e^{i(x+iy)} = 3 \Rightarrow e^{-y} e^{ix} = 3$

Since $|e^{ix}| = 1$, we have $\begin{cases} e^{-y} = 3 \\ e^{ix} = 1 \end{cases}$

real logarithm ln

$\Rightarrow y = -\mathrm{Log}\, 3, \quad x = 2k\pi, \ k \in \mathbb{Z}.$

$\therefore z = 2k\pi - i\,\mathrm{Log}\, 3$

(c) $\cos z = i \sin z \Rightarrow \dfrac{e^{iz} + e^{-iz}}{2} = i\,\dfrac{e^{iz} - e^{-iz}}{2i}$.

$\therefore e^{iz} + e^{-iz} = e^{iz} - e^{-iz} \Rightarrow 2e^{-iz} = 0 \Rightarrow e^{-iz} = 0$

no solution.

18. (a) $\lim_{z\to 0} \dfrac{\sin z}{z} = \lim_{z\to 0} \dfrac{\sin z - \sin 0}{z - 0} = (\sin z)'\Big|_{z=0} = \cos 0 = 1.$

(b) $\lim_{z\to 0} \dfrac{\cos z - 1}{z} = \lim_{z\to 0} \dfrac{\cos z - \cos 0}{z - 0} = (\cos z)'\Big|_{z=0} = -\sin 0 = 0$

§ 3.3. 4. $\mathrm{Log}\, e^{z} = \mathrm{Log}\, |e^{z}| + i\,\mathrm{Arg}\, e^{z}$

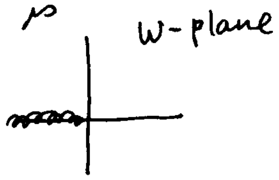
$= \mathrm{Log}\, e^{x} + i\,\mathrm{Arg}\, e^{iy}$

$= x + iy$ if and only if $-\pi < y \le \pi$

$\therefore \mathrm{Log}\, e^{z} = z. \iff -\pi < y \le \pi.$

6. $\operatorname{Log} z^2 \neq 2 \operatorname{Log} z$ (for example: $\operatorname{Log}(-1)(-1) = 0$
$\operatorname{Log}(-1) + \operatorname{Log}(-1) = \pi + \pi$)

9. Set $w = 4 + i - z$. Domain of analyticity of $\operatorname{Log} w$ is $\mathbb{C} \setminus \{w \le 0\}$

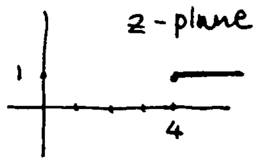


$w \le 0 \Rightarrow 4 + i - z \le 0 \Rightarrow (4-x) + i(1-y) \le 0$

$\therefore 4 - x \le 0, \quad 1 - y = 0$

$\therefore$ Domain of analyticity of $\operatorname{Log}(4+i-z)$ is

$\mathbb{C} \setminus \{x + iy : x \ge 4, y = 1\}$



11. If take the principal branch of $\log(z^2 + 2z + 3)$, the branch cut is given by

$$z^2 + 2z + 3 \le 0$$

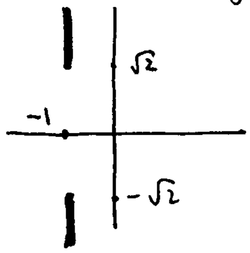
$\therefore (x+iy)^2 + 2(x+iy) + 3 \le 0 \Rightarrow (x^2 - y^2 + 2x + 3) + i(2xy + 2y) \le 0$

$\therefore \begin{cases} x^2 - y^2 + 2x + 3 \le 0 & (1) \\ 2xy + 2y = 0 & (2) \end{cases}$

$(2) \Rightarrow y = 0$ or $x = -1$.

If $y = 0$, (1) has no solution as $x^2 + 2x + 3 = (x+1)^2 + 2 > 0$

If $x = -1$, $y^2 \ge 2$. $\therefore$ domain of analyticity of $\operatorname{Log}(z^2 + 2z + 3)$ is



$\therefore \operatorname{Log}(z^2 + 2z + 3)$ is differentiable at $z = -1$.

$\therefore$ We can take the principal branch.

$$f'(z) = \frac{2z + 2}{z^2 + 2z + 3}. \qquad \therefore f'(-1) = 0.$$

13. (a) $\mathrm{Log}(2z-1)$ (b) $\mathcal{L}_0(2z-1)$ (c) $\mathcal{L}_{\frac{\pi}{2}}(2z-1)$

§ 3.5

1. (c) $2^{\pi i} = e^{\pi i \log 2} = e^{\pi i\cdot(\mathrm{Log}\, 2 + i\, 2k\pi)} = e^{-2k\pi^2} e^{i\pi \mathrm{Log}\, 2}$

3. (b) i^{2i} : principal value is $e^{2i\,\mathrm{Log}\, i} = e^{2i(\mathrm{Log}|i| + i\,\mathrm{Arg}\, i)}$

$= e^{2i\cdot i\frac{\pi}{2}} = e^{-\pi}$

6. (b) $z^\alpha z^\beta = e^{\alpha\,\mathrm{Log}\, z}\cdot e^{\beta\,\mathrm{Log}\, z} = e^{(\alpha+\beta)\mathrm{Log}\, z} = z^{\alpha+\beta}$

principal branch of z^α and prin. branch of z^β — principal branch of $z^{\alpha+\beta}$

7. prin. branch of $z^{1+i} = e^{(1+i)\mathrm{Log}\, z}$

its derivative is : $(1+i)\, z^{(1+i)-1} = (1+i)z^i$ (using principal branch)

$= (1+i)\, e^{i\,\mathrm{Log}\, z}$

∴ At $z=i$, $(1+i)\, e^{i\,\mathrm{Log}\, i} = (1+i)\, e^{i\cdot\frac{\pi}{2}i} = (1+i)e^{-\frac{\pi}{2}}$

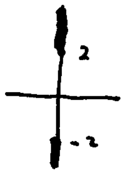
Extra Problem. $(4+z^2)^{\frac12}$, set $w = 4+z^2$.

(a) Consider the principal branch of $w^{\frac12}$. $w\le 0 \Rightarrow$

$4+z^2\le 0 \Rightarrow 4+x^2-y^2+2xyi \le 0$

$\therefore \begin{cases} 4+x^2-y^2\le 0 & (1) \\ 2xy = 0 & (2) \end{cases}$ (2) $\Rightarrow x=0$ or $y=0$

If $x=0$, then $y^2\ge 4$ ∴ $y\ge 2$ or $y\le -2$.



If $y=0$, no solution to (1).

∴ principal branch of $(4+z^2)^{\frac12}$ is analytic in $\mathbb{C}\setminus\{iy : |y|\ge 2\}$

(b) $w = 4+z^2 = z^2\cdot\left(\frac{4}{z^2}+1\right)$

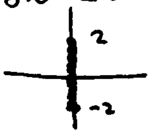
$\therefore w^{\frac12} = e^{\frac12\log[z^2\cdot(\frac{4}{z^2}+1)]} = e^{\frac12\log z^2 + \frac12\log(\frac{4}{z^2}+1)} = e^{\log z + \frac12\log(\frac{4}{z^2}+1)}$

$= z\cdot e^{\frac12\log(\frac{4}{z^2}+1)}$. Consider the principal branch:

$\frac{4}{z^2}+1\le 0 \Rightarrow \frac{4}{(x+iy)^2}+1\le 0 \Rightarrow 1+\frac{4(x-iy)^2}{(x^2+y^2)^2}\le 0$

$\Rightarrow (x^2+y^2)^2+4(x^2-y^2)-8xyi\le 0$. ∴ $xy=0$. If $x=0$ $y^2\le 4$

If $y=0$, $x=0$. ∴



the branch: $z\, e^{\frac12\mathrm{Log}(1+\frac{4}{z^2})}$ works