Solutions to HW4 (Math 300)

(1): For what values of z are the functions below differentiable? analytic? Justify your answers.

(a) $f(x) = z(\text{Im}(z))$

Solution: In terms of x, y, we can write $f(z) = (x + iy)y = xy + iy^2$. So, $u(x, y) = xy$ and $v(x, y) = y^2$. The partial derivatives are:

$$\begin{aligned} u_x &= y \quad v_x = 0 \\ u_y &= x \quad v_y = 2y \end{aligned}$$

The Cauchy-Riemann equations become:

$$\begin{aligned} y &= 2y \\ x &= -0 \end{aligned}$$

Thus, C-R hold only at the origin. Since the partial derivatives exist in a neibhbourhood of the origin and are continuous at the origin (in fact, they are continuous everywhere), f is differentiable at the origin and nowhere else. And f is analytic nowhere.

(b) $f(x) = (\text{Re}(z))^2 + (\text{Im}(z))^2$

Solution: In terms of x, y, we can write $f(z) = x^2 + y^2$. So, $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. The partial derivatives are:

$$\begin{aligned} u_x &= 2x \quad v_x = 0 \\ u_y &= 2y \quad v_y = 0 \end{aligned}$$

The Cauchy-Riemann equations become:

$$\begin{aligned} 2x &= 0 \\ 2y &= -0 \end{aligned}$$

Thus, C-R hold only at the origin. Since the partial derivatives exist in a neibhbourhood of the origin and are continuous at the origin (in fact, they are continuous everywhere), f is differentiable at the origin and nowhere else. And f is analytic nowhere.

(2) Section 2.4:

(a) #8: Show that if f is analytic in a domain D and either $\text{Re}f(z)$ or $\text{Im}f(z)$ is constant in D, then $f(z)$ must be constant in D.

Solution: Without loss of generality, we may assume that $\text{Re}f(z)$ is constant in D. Write $f = u+iv$ (so, $u = \text{Re}f(z)$ and $v = \text{Im}f(z)$). Thus, $u_x = 0$ and $u_y = 0$. By C-R equations, we have $v_x = 0$ and $v_y = 0$. Now, apply Theorem 1 on page 40 to see that v is constant in D and hence f is constant in D.

(b): #14: Show that if the analytic function $w = f(z)$ maps a domain D onto a portion of a straight line, then f must be constant throughout D.

Solution: In the $w = (u, v)$-plane a straight line may be written $v = au+b$ or $u =$ constant. The latter can be treated the same way as the former with $a = 0$, reversing the roles of u and v. So, we will assume that f maps D into the line $v = au + b$. But then

$$v_x = au_x \tag{1}$$

and

$$v_y = au_y \tag{2}$$

Using (1) and the C-R equation $u_x = v_y$, we get $v_x = av_y$. Into this equation, we plug (2) and obtain $v_x = a^2 u_y$. Into this equation, plug the C-R equation $u_y = -v_x$, and we obtain $v_x = -a^2 v_x$.

If $a \neq 0$, then $v_x = 0$ and thus by the preceding equations, all partial derivatives of u and v are 0. Thus, by Theorem 1 on page 40 both u and v are constant in D and hence f is constant in D.

If $a = 0$, then by (1), again $v_x = 0$ and we reach the same conclusion.

(3) Section 2.5:

(a) #2: Find the most general harmonic polynomial of the form $u(x, y) = ax^2 + bxy + cy^2$.

Solution: The Laplacian of u is $2a + 2c$. Thus u will be harmonic if and only if $c = -a$. So, the most general form is $u(x, y) = ax^2 + bxy - ay^2$.

(b) #6: Show that if v is a harmonic conjugate of u in a domain D, then uv is harmonic.

Solution: The Laplacian of uv is

$$(u_x v + uv_x)_x + (u_y v + uv_y)_y = u_{xx}v + 2u_x v_x + uv_{xx} + u_{yy}v + 2u_y v_y + uv_{yy}$$

$$= (u_{xx} + u_{yy})v + 2(u_x v_x + u_y v_y) + u(v_{xx} + v_{yy}).$$

The first term vanishes since u is harmonic. The last term vanishes since v is harmonic. The middle term vanishes by the C-R equations: $u_x v_x + u_y v_y = -v_y u_y + u_y v_y = 0$. Thus the Laplacian of uv vanishes. The 2nd order partials of u and v are continuous and thus so are the 2nd order partials of uv. Thus, uv is harmonic.

(4) For each of the following functions, decide whether or not u(x, y) is harmonic in the entire plane and if so, find a harmonic conjugate.

(a) $u(x, y) = 2x - x^3 + 3xy^2$

Solution: $u_x = 2 - 3x^2 + 3y^2$, and so $u_{xx} = -6x$. And $u_y = 6xy$, and so $u_{yy} = 6x$. Thus, $u_{xx} + u_{yy} = 0$, and so u satisfies Laplace's equation. Since all of its 2nd order partials are continuous, it is harmonic.

To find a harmonic conjugate: let

$$v(x, y) = \int v_x dx = -\int u_y dx = -\int 6xy dx = -3x^2 y + \psi(y)$$

Thus, $v_y = -3x^2 + \psi'(y)$. By C-R, $v_y = u_x = 2 - 3x^2 + 3y^2$. Equating these last two expressions we see that $\psi'(y) = 2 + 3y^2$ and so $\psi(y) = 2y + y^3 + c$, where c is a real constant. Thus, the general form of a harmonic conjugate is $v(x, y) = -3x^2 y + 2y + y^3 + c$.

(b) $u(x, y) = 3x - x^3 + 3xy^2$

Solution: The difference between the u in part (a) and the u in part (b) is x, whose second derivative vanishes. Thus, this u is also harmonic. Proceeding as in (a), we find that the general form of a harmonic conjugate is $v(x, y) = -3x^2 y + 3y + y^3 c$.

(c) $u(x; y) = 3x - 2x^3 + 3xy^2$

Solution: The difference between the u in part (a) and the u in part (c) is $x - x^3$, whose 2nd partial with respect is $6x$ which does not vanish in the entire plane. Thus, this u cannot be harmonic.

(5) Which of the following polynomials can be factored into linear and quadratic factors, each with real coefficients? For each that can be so factored, find the factors.

(a) $p(z) = z^3 + z - 2$

Solution:

Since p has real coefficients, it can be factored into linear and quadratic factors, each with real coefficients (as in Example 2, page 101). By inspection $z = 1$ is a zero of $p(z)$. Dividing $p(z)$ by $z - 1$, we find that $p(z) = (z - 1)(z^2 + z + 2)$, giving the desired factorization (the quadratic formula shows that this quadratic factor has no real roots, and so there can be no further factorization). So, we obtain a factorization into a linear factor and a quadratic factor, each with real coefficients.

(b) $p(z) = (z - i)^2 + 2iz$
Solution: Expanding, we see that $p(z) = z^2 - 2iz + (-i)^2 + 2iz = z^2 - 1$, which factors as $(z - 1)(z + 1)$, as a product of two linear factors.

(6) Find the Taylor form of the following polynomials centered at z_0:
(a) $p(z) = iz^3 - 5z + 2, z_0 = i$
Solution: We wish to re-write $p(z)$ in the form

$$d_0+d_1(z-i)+d_2(z-i)^2+d_3(z-i)^3 = (d_0-id_1-d_2+d_3i)+(d_1-2id_2-3d_3)z+(d_2-3id_3)z^2+d_3z^3$$

. Comparing this form with the original expression for $p(z)$, we see that

$$d_3 = -i, d_2 - 3id_3 = 0, d_1 - 2id_2 - 3d_3 = -5, d_0 - id_1 - d_2 + d_3i = 2$$

Thus, $d_3 = i$ and then $d_2 = 3id_3 = 3i^2 = -3$ and continuing, we find $d_1 = -5 - 3i$ and $d_0 = 3 - 5i$. Thus, the Taylor form for $p(z)$ is:

$$(3 - 5i) + (-5 - 3i)(z - i) - 3(z - i)^2 + i(z - i)^3.$$

Alternatively, one can use the formulas on page 103.

(b) $p(z) = 3i, z_0 = 7 + 5i$.
Solution: The Taylor form for a constant is that constant (for any z_0). So, the Taylor form is $3i$.