Solutions to HW2 (Math 300)

(1):

Section 1.4:

2b:

$$2e^{3+i\pi/6} = 2e^3(\cos(\pi/6) + i\sin(\pi/6)) = e^3\sqrt{3} + ie^3$$

2c:

$$e^{4e^{i\pi/3}} = e^{4(\cos(\pi/3)+i\sin(\pi/3))} = e^2 e^{4i\sqrt{3}/2} = e^2\cos(2\sqrt{3}) + ie^2\sin(2\sqrt{3})$$

4b:

$$\frac{2+2i}{-\sqrt{3}+i} = \frac{2\sqrt{2}e^{i\pi/4}}{2e^{i5\pi/6}} = \sqrt{2}e^{i(\pi/4-5\pi/6)} = \sqrt{2}e^{i(-7\pi/12)}$$

4c:

$$\frac{2i}{3e^{4+i}} = \frac{2e^{i\pi/2}}{3e^{4+i}} = \frac{2}{3e^4}e^{i(\pi/2-1)}$$

12a: By DeMoivre's Theorem and the binomial theorem,

$$\cos(3\theta)+i\sin(3\theta)) = (\cos(\theta)+i\sin(\theta))^3 = \cos^3(\theta)+3i\cos^2(\theta)\sin(\theta)-3\cos(\theta)\sin^2(\theta)-i\sin^3(\theta)$$

The imaginary part of the R.H.S. is $\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$. Thus,

$$\sin(3\theta)) = \cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

15a: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$e^{z_1}e^{z_2} = e^{x_1}(\cos(y_1) + i\sin(y_1))e^{x_2}(\cos(y_2) + i\sin(y_2)) =$$

$$e^{x_1}e^{x_2}(\cos(y_1)+i\sin(y_1))(\cos(y_2)+i\sin(y_2)) = e^{x_1+x_2}(\cos(y_1+y_2)+i\sin(y_1+y_2))$$

the latter equality by the ordinary law of exponents for the real exponential and the formula for the sin and cos of the sum of two angles as in eqn (7), p. 18.

15b: By 15a, $e^{z_1} = e^{z_2}e^{z_1-z_2}$. Now, divide both sides by e^{z_2}.

(2):

$$(1-i)^{1/2} = (\sqrt{2}e^{-i\pi/4+2k\pi})^{1/2} = (2^{1/4}e^{-i\pi/8+k\pi}) = 2^{1/4}e^{-i\pi/8} \text{ and } 2^{1/4}e^{i7\pi/8}$$

$$((1-i)^{3/2} = ((1-i)^3)^{1/2} = (\sqrt{2^3}e^{-i3\pi/4+2k\pi})^{1/2} = (2^{3/4}e^{-i3\pi/8+k\pi}) = 2^{3/4}e^{-i3\pi/8} \text{ and } 2^{3/4}e^{i5\pi/8}$$

(3):

Section 1.6: (see sketch file)

a: not open and not a domain
c: open and a domain
f: open, but not a domain.

(4): (see sketch file)

a: $f(D) = \{z : 1/3 < |z| < 1/2\}$
b: $f(D) = \{z : -\pi/2 < \text{Arg}(z) < -\pi/4\}$

(5):

a: $f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. So, the real part is $u(x, y) = x^3 - 3xy^2$ and the imaginary part is $v(x, y) = 3x^2y - y^3$.

b:

$$f(z) = \frac{e^z}{z} = \frac{e^x(\cos(y) + i\sin(y))}{x + iy} = \frac{e^x(\cos(y) + i\sin(y))(x - iy)}{(x + iy)(x - iy)}$$

$$= \frac{(xe^x \cos(y) + e^x y \sin(y)) + i(xe^x \sin(y) - ye^x \cos(y))}{x^2 + y^2}$$

So, the real part is:

$$\frac{xe^x \cos(y) + e^x y \sin(y)}{x^2 + y^2}$$

and the imaginary part is:

$$\frac{e^x x \sin(y) - ye^x \cos(y)}{x^2 + y^2}$$