§ 6.1. 1. (f) $\sin\left(\frac{1}{3z}\right)$. has singularity at 0. (and only at 0)

Near 0, $\sin\left(\frac{1}{3z}\right) = \frac{1}{3z} - \frac{1}{3!}\left(\frac{1}{3z}\right)^3 + \frac{1}{5!}\left(\frac{1}{3z}\right)^5 - \cdots$

$\therefore$ $\operatorname{Res}(0) = a_{-1} = \frac{1}{3}$

(i) $\sqrt{z}$ (taking the principal branch)

$= e^{\frac{1}{2}\operatorname{Log} z} = e^{\frac{1}{2}\operatorname{Log}|z| + \frac{1}{2} i \operatorname{Arg} z}$

If $\sqrt{z} = 1$, then $\frac{1}{2}\operatorname{Log}|z| + \frac{1}{2} i \operatorname{Arg} z = 2k\pi i$

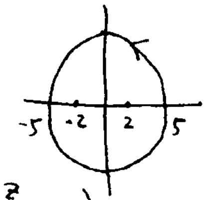
so, $|z| = 1, \Rightarrow k=0, \quad z = 1$.

Note 1 is not a zero of z^2.

$(1-\sqrt{z})' = -\frac{1}{2\sqrt{z}}$ which is not 0 at $z=1$

$\therefore$ 1 is a simple pole of $z^2/(1-\sqrt{z})$, $\operatorname{Res}(1) = \left.\frac{z^2}{(1-\sqrt{z})'}\right|_{z=1} = \frac{1}{-\frac{1}{2}} = -2$

3 (a) $\displaystyle\oint_{|z|=5} \frac{\sin z}{z^2-4}\,dz = \oint_{|z|=5} \frac{1}{4}\left(\frac{\sin z}{z-2} - \frac{\sin z}{z+2}\right)dz$



Residue Thm $\frac{1}{4}2\pi i \operatorname{Res}\left(\frac{\sin z}{z-2}; 2\right) - \frac{1}{4}2\pi i \operatorname{Res}\left(\frac{\sin z}{z+2}; -2\right)$

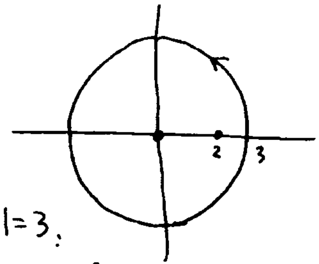
$= \frac{\pi i}{2}\sin 2 - \frac{\pi i}{2}\sin(-2) = \pi i \cdot \sin 2$

(Note: 2 is a simple pole of $\frac{\sin z}{z-2}$

-2 . . . . $\frac{\sin z}{z+2}$)

(Note: One can compute the residues of $\frac{\sin z}{z^2-4}$ at ± 2 to solve the problem, without using the partial fraction)

(b) $\displaystyle\oint_{|z|=3} \frac{e^z}{z(z-2)^3}\,dz$.



$\frac{e^z}{z(z-2)^3}$ has singularities within $|z|=3$.

at $z=0$, single pole, $\operatorname{Res}(0) = \left.\frac{e^z}{(z(z-2)^3)'}\right|_{z=0}$

$= \frac{1}{-8} = -\frac{1}{8}$

at $z=2$, pole of order 3.

$\operatorname{Res}(2) = \frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}\left[(z-2)^3 \cdot \frac{e^z}{z(z-2)^3}\right] = \frac{1}{2}\lim_{z\to 2}\left(\frac{e^z}{z}\right)''$

$= \frac{1}{2}\lim_{z\to 2}\left[\frac{e^z}{z} - \frac{e^z}{z^2}\right]' = \frac{1}{2}\lim_{z\to 2}\left[\frac{e^z}{z} - 2\frac{e^z}{z^2} + 2\frac{e^z}{z^3}\right] = \frac{e^2}{8}$

Residue Theorem $\Rightarrow$ $\displaystyle\oint_{|z|=3}\frac{e^z}{z(z-2)^3}\,dz = 2\pi i\left(-\frac{1}{8}\right) + 2\pi i\left(\frac{e^2}{8}\right) = \frac{\pi i}{4}(e^2-1)$

(c) singularities of $\tan z$: $\cos z = 0 \Rightarrow \dfrac{e^{iz}+e^{-iz}}{2} = 0$

$\therefore \quad e^{2iz} = -1 \quad \therefore \quad 2iz = (\pi + 2k\pi)\, i \Rightarrow z = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$

Inside $|z| < 2\pi$, singularities are: $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

each of them is a simple pole because:

$$(\cos z)'\Big|_{\frac{\pi}{2}+k\pi} = -\sin\left(\tfrac{\pi}{2}+k\pi\right) \neq 0$$

$$\therefore \quad \mathrm{Res}\left(\tan z; \tfrac{\pi}{2}+k\pi\right) = \frac{\sin\left(\frac{\pi}{2}+k\pi\right)}{-\sin\left(\frac{\pi}{2}+k\pi\right)} = -1$$

Cauchy Residue Theorem $\Rightarrow$

$$\int_{|z|<2\pi} \tan z \, dz = 2\pi i\left(\mathrm{Res}\left(\tfrac{\pi}{2}\right) + \mathrm{Res}\left(-\tfrac{\pi}{2}\right) + \mathrm{Res}\left(\tfrac{3\pi}{2}\right) + \mathrm{Res}\left(-\tfrac{3\pi}{2}\right)\right)$$
$$= -8\pi i.$$

6. Since f is analytic and has a zero of order m at z_0, we can write $f(z) = (z-z_0)^m g(z)$, $g(z_0) \neq 0$, g analytic

(Theorem 16, p. 278)

$$\therefore \quad \frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)}$$

$$= \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

As $g'(z)/g(z)$ is analytic at z_0, f'/f has a simple pole at z_0

$\mathrm{Res}\left(\frac{f'}{f}, z_0\right) = m.$

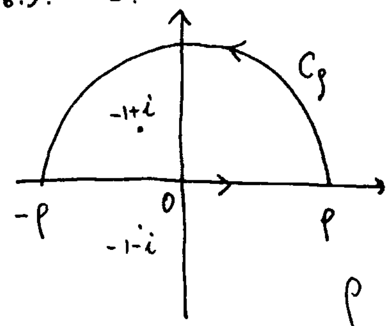
4. Near z_0,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$f'(z) = \cdots - \frac{2a_{-2}}{(z-z_0)^3} - \frac{a_{-1}}{(z-z_0)^2} + a_1 + 2a_2(z-z_0) + \cdots$$

the coefficient for $\frac{1}{z-z_0}$ in the Laurent series of $f'(z)$ is 0, $\therefore$ $\operatorname{Res}(f'; z_0) = 0$.

§6.3. 1.



Consider $\displaystyle\int_{C_\rho \cup [-\rho,\rho]} \frac{dz}{z^2+2z+2}$

$\frac{1}{z^2+2z+2}$ has two poles: $-1+i$, $-1-i$, both are single.

$$\int_{C_\rho\cup[-\rho,\rho]} \frac{dz}{z^2+2z+2} \overset{\text{Res. Thm}}{=\!=} 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}; -1+i\right)$$

$$= 2\pi i \cdot \frac{1}{(z^2+2z+2)'}\Big|_{i-1} = 2\pi i \cdot \frac{1}{2i} = \pi$$

degree of $z^2+2z+2 = 2 \geq 0 =$ degree of 1 (constant polynomial)

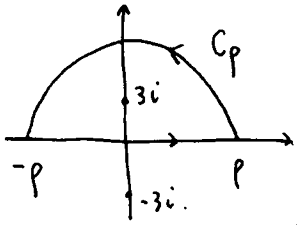
Therefore, by result proved in book,

$$\lim_{\rho\to\infty} \int_{C_\rho} \frac{dz}{z^2+2z+2} = 0$$

Along $[-\rho, \rho]$, $\frac{dz}{z^2+2z+2} = \frac{dx}{x^2+2z+2}$

$\therefore$ p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$.

2.



Consider $\displaystyle\int_{C_\rho\cup[-\rho,\rho]} \frac{z^2\,dz}{(z^2+9)^2}$, inside the loop, $3i$ is the only singularity. $3i$ is a pole of order 2.

$$\operatorname{Res}\left(\frac{z^2}{(z^2+9)^2}; 3i\right) = \frac{1}{1!}\lim_{z\to 3i}\left[(z-3i)^2\frac{z^2}{(z^2+9)^2}\right]' = \lim_{z\to 3i}\left[\frac{z^2}{(z+3i)^2}\right]'$$

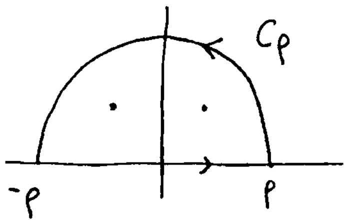
$$= \frac{2z}{(z+3i)^2}\Big|_{z=3i} - \frac{2z^2}{(z+3i)^3}\Big|_{z=3i} = \frac{1}{2}\cdot\frac{1}{6i}.$$

degree of $(z^2+9)^2 = 4 \geq 2 =$ deg of z^2, $\therefore \displaystyle\int_{C_\rho}\frac{z^2dz}{(z^2+9)^2} \to 0$ as $\rho\to\infty$

$\therefore$ p.v. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+9)^2} = 2\pi i\cdot\frac{1}{2\cdot 6i} = \frac{\pi}{6}$.

3. Note: $\displaystyle\int_{-\rho}^{\rho} \frac{x^2+1}{x^4+1}\,dx = 2\int_{0}^{\rho} \frac{x^2+1}{x^4+1}\,dx$

Consider $\displaystyle\int_{C_\rho \cup [-\rho,\rho]} \frac{z^2+1}{z^4+1}\,dz$



There're 4 zeros for z^4+1, but only $\frac{1}{\sqrt2}+\frac{i}{\sqrt2}$, $-\frac{1}{\sqrt2}+\frac{i}{\sqrt2}$ are inside the loop, and they are single poles

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1};\ \frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right) = \left.\frac{z^2+1}{4z^3}\right|_{\frac{1}{\sqrt2}+\frac{i}{\sqrt2}} = \frac{1}{2\sqrt2 i}$$

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1};\ -\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right) = \left.\frac{z^2+1}{4z^3}\right|_{-\frac{1}{\sqrt2}+\frac{i}{\sqrt2}} = \frac{1}{2\sqrt2 i}$$

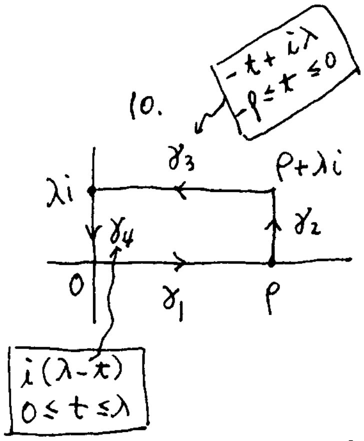
degree of $z^4+1 = 4 \geq 2 =$ degree of z^2+1.

$\therefore \displaystyle\int_{C_\rho} \frac{z^2+1}{z^4+1}\,dz \longrightarrow 0$ as $\rho \to \infty$

$\therefore$ Residue Theorem $\Rightarrow$ p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{z^2+1}{z^4+1}\,dz = 2\pi i\left(\frac{1}{2\sqrt2 i}+\frac{1}{2\sqrt2 i}\right) = \frac{2\pi}{\sqrt2}$

$$\therefore \int_0^\infty \frac{x^2+1}{x^4+1}\,dx = \frac{\pi}{\sqrt2}$$

10.



$\gamma_\rho = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, e^{-z^2} is entire $\therefore \displaystyle\int_{\gamma_\rho} e^{-z^2}\,dz = 0$ (1)

$\displaystyle\lim_{\rho\to\infty}\int_{\gamma_1} e^{-z^2}\,dz = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt\pi}{2}$ (given) (2)

$$\lim_{\rho\to\infty}\left|\int_{\gamma_2} e^{-z^2}\,dz\right| = \lim_{\rho\to\infty}\left|\int_0^\lambda e^{-(\rho+it)^2}\,i\,dt\right| \le \lim_{\rho\to\infty}\int_0^\lambda \left|e^{-\rho^2+t^2-2\rho t i}\right|dt$$

$$\le \lim_{\rho\to\infty}\int_0^\lambda e^{-\rho^2+\lambda^2}\,dt = \lim_{\rho\to\infty} e^{-\rho^2+\lambda^2}\cdot\lambda = 0 \quad (3)$$

$$\lim_{\rho\to\infty}\int_{\gamma_3} e^{-z^2}\,dz = \lim_{\rho\to\infty}\int_{-\rho}^0 e^{-(-t+i\lambda)^2}(-1)\,dt = -e^{\lambda^2}\lim_{\rho\to\infty}\int_{-\rho}^0 e^{-t^2+2i\lambda t}\,dt$$

$$\underset{s=-t}{=} -e^{\lambda^2}\lim_{\rho\to\infty}\int_\rho^0 e^{-s^2-2is\lambda}(-ds) = -e^{\lambda^2}\int_0^\infty e^{-s^2-2s\lambda i}\,ds$$

$$= -e^{\lambda^2}\int_0^\infty\left[e^{-s^2}\cos(2\lambda s) - i e^{-s^2}\sin(2\lambda s)\right]ds \quad (4)$$

$$\lim_{\rho\to\infty}\int_{\gamma_4} e^{-z^2}\,dz \underset{z=i(\lambda-t)}{=} \int_0^\lambda e^{(\lambda-t)^2}(-i)\,dt = -i\int_0^\lambda e^{(\lambda-t)^2}\,dt \overset{y=\lambda-t}{=} -i\int_0^\lambda e^{y^2}\,dy \quad (5)$$

①, ②, ③, ④, ⑤ $\Rightarrow$

$$0 = \frac{\sqrt{\pi}}{2} - e^{\lambda^2}\int_0^\infty \left[e^{-x^2}\cos(2\lambda x) - ie^{-x^2}\sin(2\lambda x)\right]dx - i\int_0^\lambda e^{x^2}dx \quad ⑥$$

Take the real part:

$$0 = \frac{\sqrt{\pi}}{2} - e^{\lambda^2}\int_0^\infty e^{-x^2}\cos(2\lambda x)\,dx \Rightarrow \int_0^\infty e^{-x^2}\cos(2\lambda x)\,dx = \frac{\sqrt{\pi}}{2}e^{-\lambda^2}$$

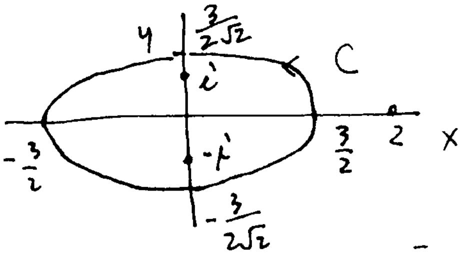
Take the imaginary part:

$$0 = e^{\lambda^2}\int_0^\infty e^{-x^2}\sin(2\lambda x)\,dx - \int_0^\lambda e^{x^2}dx$$

So, $\int_0^\infty e^{-x^2}\sin(2\lambda x)\,dx = e^{-\lambda^2}\int_0^\lambda e^{x^2}dx.$

Extra problem:

$$C: \frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{\left(\frac{3}{2\sqrt{2}}\right)^2} = 1$$



(1) $\frac{\sin z}{(z-2)^2}$ has a singularity at $z=2$ which is outside of C

$$\therefore \oint_C \frac{\sin z}{(z-2)^2}dz = 0$$

(2) $\frac{e^{2z}}{1+z^2}$ has simple poles at $i, -i$ which are inside C

$$\text{Res}(i) = \frac{e^{2i}}{2i} = \frac{\cos 2 + i\sin 2}{2i} = -\frac{1}{2}\cos 2 + \frac{1}{2}\sin 2$$

$$\text{Res}(-i) = \frac{e^{-2i}}{-2i} = \frac{\cos 2 - i\sin 2}{-2i} = \frac{1}{2}\cos 2 + \frac{1}{2}\sin 2$$

$\therefore$ Residue Theorem $\Rightarrow$

$$\oint_C \frac{e^{2z}}{1+z^2}dz = 2\pi i\left(\text{Res}(i) + \text{Res}(-i)\right)$$
$$= 2\pi i \cdot \sin 2$$