# Homework #10 SOLUTIONS

**Each numbered question is of equal value.**

(1) §5.5 #4.

The Taylor series for $\sin(z) = \sum_{j=0}^{\infty}(-1)^j(z)^{2j+1}/(2j+1)!$ is valid for all $z$. Thus, for $z \neq 0$,

$$\frac{\sin(2z)}{z^3} = \frac{\sum_{j=0}^{\infty}(-1)^j(2z)^{2j+1}/(2j+1)!}{z^3} =$$

$$\sum_{j=0}^{\infty}(-1)^j 2^{2j+1} z^{2j-2}/(2j+1)! = 2z^{-2} - (8/(3!)) + (32/5!)z^2 - \ldots$$

This expansion is valid for all $z \neq 0$, and thus, by uniqueness (essentially Theorem 15) this is the Laurent series for $(\sin(2z))/z^3$ in the domain $\mathbb{C} \setminus \{0\}$.

(2) Find the Laurent series for $f(z) = \frac{1}{1-z}$ valid centered at $z_0 = i$ in the domain $|z - i| > \sqrt{2}$.

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{-1}{(z-i)}\frac{1}{(1-\frac{1-i}{z-i})} =$$

$$\frac{-1}{(z-i)}\sum_{j=0}^{\infty}(\frac{1-i}{z-i})^j = -\sum_{j=1}^{\infty}(1-i)^{j-1}(z-i)^{-j}$$

which is valid in the domain $D: \ |z-i| > \sqrt{2}$ since $|1-i| = \sqrt{2}$. Thus, by uniqueness, this is the Laurent series expansion for $f(z) = \frac{1}{1-z}$ in $D$.

(3) Find two different Laurent series for $f(z) = \frac{3}{z+z^3}$ centered at $z_0 = 0$.

The singularities are at 0, $i$ and $-i$. Let $D = \{0 < |z| < 1\}$. Then $f(z)$ is analytic in $D$ and in this domain, we have

$$\frac{3}{z+z^3} = \frac{1}{z}(\frac{3}{1+z^2}) = \frac{3}{z}(\sum_{j=0}^{\infty}(-z^2)^j) = 3\sum_{j=0}^{\infty}(-1)^j z^{2j-1}$$

(note that this is valid since $|z| < 1$ in $D$). So, the latter expression is the Laurent series for $f(z)$ in $D$.

Now, let Let $D' = \{|z| > 1\}$. Then $f(z)$ is analytic in $D'$ and in this domain, we have

$$\frac{3}{z+z^3} = \frac{3}{z^3}(\frac{1}{1+z^{-2}}) = \frac{3}{z^3}(\sum_{j=0}^{\infty}(-z^{-2})^j) = 3\sum_{j=0}^{\infty}(-1)^j z^{-2j-3}.$$

(note that this is valid since $|z| > 1$ in $D'$). So, the latter expression is the Laurent series for $f(z)$ in $D'$.

(4) §5.5 #13.

We have the Laurent coefficient:

$$a_j = (1/(2\pi i)) \int_C \frac{f(w)}{(w - z_0)^{j+1}} dw.$$

where $C$ is a positively oriented circle of radius $\rho$ such that $r < \rho < R$. Thus,

$$|a_j| \leq (1/(2\pi)) \frac{M}{\rho^{j+1}} (2\pi\rho).$$

where $M$ is the upper bound on $f$ in the annulus. Thus,

$$|a_j| \leq \frac{M}{\rho^j}$$

Let $j \geq 0$. Let $\rho \to R$, and we get

$$|a_j| \leq \frac{M}{R^j}$$

And letting $\rho \to r$, we get

$$|a_{-j}| \leq \frac{M}{r^{-j}} = Mr^j.$$

(5) §5.6 #2.

$$f(z) = \frac{1}{(2\cos(z) - 2 + z^2)^2} = \frac{1}{(2(1 - z^2/2! + z^4/4! - \dots) - 2 + z^2)^2}$$

$$= \frac{1}{((2/4!)z^4 - (2/6!)z^6 + \dots)^2} = \frac{1}{z^8} \frac{1}{((2/4!) - (2/6!)z^2 \cdots +)^2}$$

This expression is of the form $\frac{g(z)}{z^8}$ where

$$g(z) = \frac{1}{((2/4!) - (2/6!)z^2 + \dots)^2}$$

which is analytic and non-zero at $z_0 = 0$. Thus, $f(z)$ has a pole of order 8 at $z_0 = 0$.

(6) §5.6 #12.

If $f(z)$ has a pole of order $m$ at $z_0$, then

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where $g(z)$ is analytic and non-zero at $z_0$. Taking the derivative, using the quotient rule, we have

$$f'(z) = \frac{(z - z_0)^m g'(z) - m(z - z_0)^{m-1} g(z)}{(z - z_0)^{2m}} = \frac{g'(z)(z - z_0) - mg(z)}{(z - z_0)^{m+1}}$$

Thus,

$$\frac{f'(z)}{f(z)} = \frac{(z - z_0)g'(z)/g(z) - m}{z - z_0}$$

The numerator of the last experssions is analyitc at $z_0$ and has value $-m \neq 0$ at $z_0$. Thus, $z_0$ is a pole of order 1 (i.e., simple pole) for $\frac{f'(z)}{f(z)}$. Since

$$\frac{f'(z)}{f(z)} = g'(z)/g(z) - m/(z - z_0)$$

and $g'(z)/g(z)$ is analytic at $z_0$, the coefficient of $(z - z_0)^{-1}$ in its Laurent series is $-m$.

(7) Find the orders of the zeros of

(a) $\frac{2z^5 - 2z^7}{1+z}$

$f(z) = \frac{2z^5 - 2z^7}{1+z} = 2z^5(1-z)(1+z)/(1+z)$

The zeros of this function are 0 and 1 (note that the function is undefined at -1).

$f(z) = z^5(2(1-z)(1+z)/(1+z))$ where the function in parentheses is analytic and non-zero at 0. Thus, 0 is a zero of order 5.

$f(z) = (z-1)(-z^5(2(1+z)/(1+z))$ where the function in parentheses is analytic and non-zero at 1. Thus, 1 is a zero of order 1.

(b) $1 - \cos(z)$

The zeros of $f(z) = 1 - \cos(z)$ are $z_k = 2k\pi$. We have $f(z_k) = 0$ and $f'(z_k) = \sin(z_k) = 0$ and $f''(z_k) = \cos(z_k) = 1$. So, all of these zeros are of order 2.

(8) Find the isolated singularities of the following functions and classify their types (if it is a pole, find its order).

(a) $z^{-5}e^z$

The only singularity is at 0 and it is isolated. Since $e^z$ is analytic and non-zero at 0, it is a pole of order 5.

(b) $\frac{\sin z}{z+2z^2+z^3}$

The only singularities are the zeros of the denominator: 0 and -1.

For 0, we write

$$f(z) = \frac{\sin z}{z + 2z^2 + z^3} = \frac{\sin z}{z}\left(\frac{1}{(1+z)^2}\right) = (1 - z^2/3! + z^4/5! - \ldots)\left(\frac{1}{(1+z)^2}\right),$$

Since $\frac{1}{(1+z)^2}$ is analytic in the entire disk $D$ $\{|z| < 1\}$, it has a Taylor series expansion in $D$, and thus the Laurent series for $f(z)$ in the punctured disk $\{0 < |z| < 1\}$ has no negative powers. Thus, 0 is a removable singularity.

For -1, we write:

$$f(z) = \frac{\sin z}{z + 2z^2 + z^3} = \frac{\frac{\sin z}{z}}{(1+z)^2}$$

Since the numerator is analytic and non-zero at -1, we have that -1 is a pole of order 2.

(c) $\frac{e^{1/z}}{\sin(1/z)}$

The only singularities are at 0 and $z_n = 1/(n\pi))$, $n \neq 0$. Thus, each $z_n$ is an isolated singularity, but 0 is not.

$h(z) = \sin(1/z)$ is analytic at each $z = z_n$. Since $h(z_n) = \sin(n\pi) = 0$ and $h'(z_n) = -(1/z_n)^2 \cos(n\pi) \neq 0$, $h(z)$ has a simple zero at $z_n$. Thus, $1/\sin(1/z)$ has a simple pole at $z_n$. Since $e^{1/z}$ is analytic at $z_n$, $e^{1/z}/\sin(1/z)$ also has a simple pole at $z_n$.

(9) Let $f(z)$ denote the principal branch of $\mathrm{Log}(z)$. Let $g(z)$ be the Taylor series of $f(z)$ centered at $z_0 = -1 + i$.

(a) What is the radius of convergence of $g(z)$?

For $j \geq 1$, $f^{(j)}(z) = (-1)^{j-1}/z^j$. Thus, the Taylor series coefficients for $f(z)$ centered at $z_0 = -1 + i$ are

$$a_j = (-1)^{j-1}/(j(-1+i)^j)$$

By the ratio test the radius of convergeence of this series is $|-1+i| = \sqrt{2}$.

(b) What is the radius of the largest disk centered at $z_0 = -1 + i$ on which $f(z) = g(z)$.

The Taylor series of an analytic function is a valid representation on the largest disk centered at $z_0$ on which $f(z)$ is analytic. In this case, since $z_0 = -1 + i$ and -1 is the closest point to $z_0$ at which $f(z)$ is not analytic, 1 is the radius of the largest disk centered at $z_0 = -1 + i$ on which $f(z) = g(z)$.

(10) Section 5.4: 4.

$\sum_{j=1}^{\infty} z^j/j^2$ is absolutely convergent for all $|z| = 1$ since, as we know from HW9, $\sum_{j=1}^{\infty} 1/j^2$ converges. Thus, $\sum_{j=1}^{\infty} z^j/j^2$ converges for all $|z| = 1$.

$\sum_{j=1}^{\infty} z^j/j^2$ is the harmonic series when $z = 1$ and thus diverges; it is the alternating harmonic series when $z = -1$ and thus converges.

For all $|z| >= 1$, the $j$-th term of $\sum_{j=1}^{\infty} z^j$ has absolute value equal to 1 and thus cannot converge to 0. As this is a necessary condition for convergence, the series $\sum_{j=1}^{\infty} z^j$ diverges for all $|z| = 1$.