Solution

Prob 1

(a) $\exists\, x > 1$, s.t. $\forall\, y \in \mathbb{R}$, $2x + y \geq 0$

(b) $\forall\, x \in [0,1)$, $f(x) \leq 1$ and $f(x) > -1$

(c) $x \notin A$ or $x \in B$

Prob 2

(a) True. $\forall\, x \in A \cap C$, we know $x \in A$ $\because A \subseteq B$
$\therefore x \in B$. $x \in A \cap C$ $\therefore x \in C$ $\therefore x \in B \cap C$
$\therefore A \cap C \subseteq B \cap C$

(b) Converse: $A \cap C \subseteq B \cap C \implies A \subseteq B$
False. Counterexample $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$
$A \cap C = \{1\} = B \cap C$, $A \not\subseteq B$.

(c) Inverse: $A \not\subseteq B \implies A \cap C \not\subseteq B \cap C$
False. The same example as above

Prob 3.

(a) f is not surjective. For example, 2 is not in the range of f.

(b) If $f(x_1) = f(x_2)$, we have 3 cases: i) x_1, x_2 even ii) x_1 odd, x_2 odd iii) x_1 even, x_2 odd

i) $f(2n_1) = f(2n_2)$, then $n_1^2 = n_2^2 \implies n_1 = n_2$, $(n_1, n_2 > 0)$

ii) $f(2n_1 - 1) = f(2n_2 - 1)$, then $-5n_1 = -5n_2 \implies n_1 = n_2$

iii) $f(2n_1) = f(2n_2 - 1)$, then $n_1^2 = -5n_2 < 0$ can not happen.

$\therefore f$ is injective. (i.e. $f(x_1) = f(x_2) \implies x_1 = x_2$)

Prob 4.

(a) A is finite or denumerable

(b) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$
$\therefore (g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective,
we have $x_1 = x_2$ $\therefore f$ is injective
$f: A \to \mathbb{N}$ injective $\implies A$ is countable, by theorem.

(c) False. S is denumerable, so we can write S
as $S = \{\underbrace{s_1, s_2, s_3, \cdots}_{\text{infinitely many}}\}$

$S \setminus \{s_1\}$, $S \setminus \{s_1, s_2\}$, $S \setminus \{s_1, s_2, s_3\}$, $\cdots$
are denumerable subsets of S. There're infinitely many of them.