FIRST MIDTERM EXAM

Math 220, February 9, 2010

Name (Last, First): ______________________

Student Number: ______________________

There are four problems in this exam which add up to 40 points.

Problem 1. [12/40]

- Let P be the statement: $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}$, such that $n < k < n^2 + 1$.
  (1) Write done the negation of P.

  $\exists n \in \mathbb{N}, \forall k \in \mathbb{N}, \quad k \leq n$ or $k \geq n^2+1$

  (2) Is P true or false? Justify your answer.

  False. $\quad n = 1. \quad 1 < k < 1^2+1 = 2. \quad$ no k

- Let P be the statement: $\forall a \in \mathbb{Q}, \forall b \in \mathbb{Q}$, $a + (b^2 + 1)\sqrt{2}$ is irrational.
  (1) Write down the negation of P.

  $\exists a \in \mathbb{Q}, \exists b \in \mathbb{Q}, \quad a + (b^2+1)\sqrt{2}$ is rational

  (2) Is P true or false? Justify your answer.

  True. $\quad b^2+1 \neq 0$ and b^2+1 rational, $\sqrt{2}$ irrational
  $\therefore (b^2+1)\sqrt{2}$ is irrational
  a rational $\therefore a + (b^2+1)\sqrt{2}$ is irrational

- Let P be the statement: If $f(x_1) = f(x_2)$, then $x_1 = x_2$. Write down the contrapositive of P.

  If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Problem 2. [10/40] Let A, B, C, D be non-empty sets. Show $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$.

"$\subseteq$" $\quad \forall\ (x,y) \in (A\times B)\cap(C\times D)$

$(x,y) \in A\times B \Rightarrow x\in A,\ y\in B$

$(x,y) \in C\times D \Rightarrow x\in C,\ y\in D$

$\therefore\ x\in A\cap C,\ y\in B\cap D$

$\therefore\ (x,y)\in (A\cap C)\times(B\cap D)$

$\therefore\ (A\times B)\cap(C\times D) \subseteq (A\cap C)\times(B\cap D)$

"$\supseteq$" $\quad \forall\ (x,y)\in (A\cap C)\times(B\cap D)$, we have

$x\in A\cap C,\ y\in B\cap D$

$\therefore\ x\in A,\ y\in B \Rightarrow (x,y)\in A\times B$

$x\in C,\ y\in D \Rightarrow (x,y)\in C\times D$

$\therefore\ (x,y)\in (A\times B)\cap(C\times D)$

$\therefore\ (A\times B)\times(B\cap D) \supseteq (A\cap C)\times(B\cap D)$

$\therefore\ (A\times B)\times(B\cap D) = (A\cap C)\times(B\cap D)$

Problem 3. [10/40] Let $f : \mathbb{R} \to \mathbb{R} \times [0, \infty)$ be a function defined by $f(x) = (x^3 + x, x^2)$.

(1) Is f injective? Justify your answer.

If $f(x_1) = f(x_2)$, then $(x_1^3 + x_1, x_1^2) = (x_2^3 + x_2, x_2^2)$

$$\therefore \begin{cases} x_1^3 + x_1 = x_2^3 + x_2 & (1) \\ x_1^2 = x_2^2 & (2) \end{cases}$$

$(1) \Rightarrow x_1 (x_1^2 + 1) = x_2 (x_2^2 + 1)$, (3)

from (2) (3) $\Rightarrow$ $x_1 = x_2$ $\quad \therefore f$ is injective

$(x_1^2 + 1 = x_2^2 + 1 \neq 0)$

(2) Is f surjective? Justify your answer.

f is not surjective.

$(1, 0) \in \mathbb{R} \times [0, \infty)$

But if $(x^3 + x, x^2) = (1, 0)$, we have

$$\begin{cases} x^3 + x = 1 & (1) \\ x^2 = 0 & (2) \end{cases} \quad \therefore x = 0 \text{ from } (2)$$

Contradicts to (1)

$\therefore$ There is no $x \in \mathbb{R}$ with $f(x) = (1, 0)$.

Problem 4. [8/40] Let A, B be non-empty sets. Let $f : A \to \mathbb{N}$ and $g : \mathbb{N} \to B$ be functions. If $g \circ f$ is surjective, show that B is countable.

$\forall\ b \in B$, since $g \circ f$ is surjective, $\exists\ a \in A$ such that

$$g \circ f\,(a) = b$$

$\therefore\ g(f(a)) = b,\quad f(a) \in \mathbb{N}$

$\therefore\ g : \mathbb{N} \to B$ is surjective

By the theorem we proved in class, B is countable.