Solution to HW 7

13.3 (a) ϕ (b) $(0,5)$. Note: $[0,3] \cup (3,5) = [0,5)$

(c) ϕ (density of irrational numbers)

(d) ϕ

(e) ϕ, Note: $[0,2] \cap [2,4] = \{2\}$

13.5 (a) neither (b) closed (c) neither $\left(\begin{array}{l}\text{Note:}\\ \text{bd}\,\mathbb{Q} = \mathbb{R}\\ \text{int}\,\mathbb{Q} = \phi\end{array}\right)$

(d) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$. both open & closed

(e) closed

(f) open. note: $\{x : x^2 > 0\} = (-\infty, 0) \cup (0, \infty)$

13.9 (a) Denote the set of accumulation points of S by S'.

If $x \in S'$ and x is not in $\text{int}\,S$

For any neighborhood $N(x;\varepsilon)$ of x, Since $x \notin \text{int}\,S$

$N(x;\varepsilon)$ is not contained in S, $\therefore$ $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \phi$ ①

Since $x \in S'$, $N^*(x;\varepsilon) \cap S \neq \phi$, so

$N(x;\varepsilon) \cap S \neq \phi$. ② . Now ① and ② $\Rightarrow x \in \text{bd}\,S$

(b) If $x \in \text{bd}\,S$, we have 2 cases. ~~[illegible]~~.

<u>Case 1 $x \in S$</u>. If x is not ~~[illegible]~~ an accumulation pt of S, then it is an isolated pt by definition.

<u>Case 2 $x \notin S$</u> Since $x \in \text{bd}\,S$, $\forall N(x;\varepsilon)$, we have

$N(x;\varepsilon) \cap S \neq \phi$, Since $x \notin S$.

$N^*(x;\varepsilon) \cap S \neq \phi$ $\therefore$ $x \in S'$

(2)

(13.10) If x is an isolated pt of S, by definition, we have

1) $x \in S$
2) $x \notin S'$.

For any given $N(x;\varepsilon)$.

$N(x;\varepsilon) \cap S \neq \phi$ as it contains x. ..... (A)

If $N(x;\varepsilon) \cap \mathbb{R}\backslash S = \phi$, then $N(x;\varepsilon) \subset S$ and from this we claim $x \in S'$. To see this, consider an arbitrary deleted neighborhood $N^*(x;\delta)$ of x.

If $\delta \geq \varepsilon$, then $N^*(x;\delta) \cap S$ contains $\underset{\phi}{\underset{\neq}{N^*(x;\varepsilon) \cap S}}$

If $\delta < \varepsilon$, then

$$N^*(x;\delta) \subset N^*(x;\varepsilon) \subset S$$

we still have $N^*(x;\delta) \cap S \neq \phi$

$\therefore$ $x \in S'$. but this contradicts 2).

Hence $N(x;\varepsilon) \cap \mathbb{R}\backslash S \neq \phi$ .... (B)

(A), (B) $\Longrightarrow$ $x \in \text{bd}\, S$.

(13.12) $N^*(x;\varepsilon) = (x-\varepsilon, x) \cup (x, x+\varepsilon)$

Both $(x-\varepsilon, x)$ and $(x, x+\varepsilon)$ are open, hence so is their union.

(13.14) $x = \sup S$. If $\exists\, \varepsilon_0 > 0$, s.t. $N^*(x;\varepsilon_0) \cap S = \phi$

then since $x \notin S$, $N(x;\varepsilon_0) \cap S = \phi$

$\therefore$ S lies outside $(x_0-\varepsilon_0, x_0+\varepsilon_0)$.

No elements of S can be on the right side of x, because otherwise x_0 is not an upper bd of S

Now the middle point $x_0 - \frac{\varepsilon_0}{2}$ is an upper bound of S which contradicts to x is the least upper bound of S.

Note: S bounded guarantees $\sup S$ exists

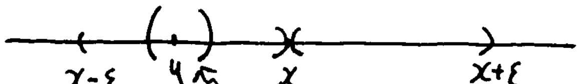
S is a infinite set makes $\sup S \notin S$ a valid assumption

13.17 We want to show: S' contains all of its accumulation points.

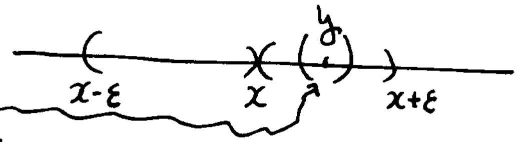
If $x \in (S')'$, then $\forall \varepsilon > 0$, $N^*(x;\varepsilon) \cap S' \neq \phi$

$\therefore \exists y \in N^*(x;\varepsilon) \cap S'$

Let's first use $y \in N^*(x;\varepsilon)$.



or



$\exists \delta > 0$, s.t. $N(y;\delta) \subset N^*(x;\varepsilon)$

Next, use $y \in S'$. $N^*(y;\delta) \cap S \neq \phi$

$\Rightarrow N^*(x;\varepsilon) \cap S \neq \phi$ $\therefore x \in S'$. $\therefore S'$ is closed by the theorem: a set is closed iff it contains all of its accumulation pts.

13.20 (a) $\mathrm{cl}\, S$ is closed $\leftarrow$ Theorem 13.17

For any closed set A, $A = \mathrm{cl}\, A$

$\therefore \mathrm{cl}\, S = \mathrm{cl}(\mathrm{cl}\, S)$.

13.21 (a) <u>Proof 1</u> $\forall x \in \mathrm{int}\, S$, $\exists N(x;\varepsilon_x) \subset S$ depends on x

$\bigcup_{x \in \mathrm{int}\, S} N(x;\varepsilon_x)$ is open

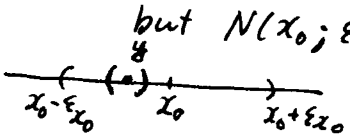
Claim $\mathrm{int}\, S = \bigcup_{x \in \mathrm{int}\, S} N(x;\varepsilon_x)$

"$\subseteq$": $\forall x_0 \in \mathrm{int}\, S$, $x_0 \in N(x;\varepsilon_x) \subseteq \bigcup_{x \in \mathrm{int}\, S} N(x;\varepsilon_x)$

"$\supseteq$": $\forall y \in \bigcup_{x \in \mathrm{int}\, S} N(x;\varepsilon_x) \Rightarrow y \in$ some $N(x_0;\varepsilon_{x_0})$, $x_0 \in \mathrm{int}\, S$

but $N(x_0;\varepsilon_{x_0}) \subset S$

$\therefore y \in \mathrm{int}\, S$



Proof 2. Show: $\mathbb{R} \setminus \operatorname{int} S$ is closed

It suffices to prove $\operatorname{bd}(\mathbb{R} \setminus \operatorname{int} S) \subseteq \mathbb{R} \setminus \operatorname{int} S$

Let $x \in \operatorname{bd}(\mathbb{R} \setminus \operatorname{int} S)$. $\therefore$ $\forall \varepsilon > 0$, we have

$$N(x;\varepsilon) \cap (\mathbb{R} \setminus \operatorname{int} S) \neq \phi \quad \cdots\cdots \text{A}$$

$$N(x;\varepsilon) \cap \Big(\underbrace{\mathbb{R} \setminus (\mathbb{R} \setminus \operatorname{int} S)}_{\operatorname{int} S}\Big) \neq \phi$$

If $x \notin \mathbb{R} \setminus \operatorname{int} S$, then $x \in \operatorname{int} S$

$\Rightarrow$ $\exists\, \varepsilon_0 > 0$, s.t. $N(x;\varepsilon_0) \subset S$

$\Rightarrow$ $N(x;\varepsilon_0) \subset \operatorname{int} S$ $\Rightarrow$ $N(x;\varepsilon_0) \cap (\mathbb{R} \setminus \operatorname{int} S) = \phi$

Contradicts to A

$\therefore$ $x \in \mathbb{R} \setminus \operatorname{int} S$ $\quad$ $\therefore$ $\operatorname{bd}(\mathbb{R} \setminus \operatorname{int} S) \subseteq \mathbb{R} \setminus \operatorname{int} S$

$\therefore$ $\mathbb{R} \setminus \operatorname{int} S$ is closed

$\therefore$ $\operatorname{int} S$ is open