Solution to HW Set 3

5.2 (a) false. take $A = \{1\}$, $B = \{2\}$

(b) True. definition of $A \cup B$

(c) false. $x \in A \setminus B$ means $x \in A$ and $x \notin B$

(d) False. If $S = \phi$ then $S \subseteq T$ always holds. So in proving $S \subseteq T$, we can begin with "Let $x \in S$".

5.4 (a) $\{2, 4\}$. (b) $\{1, 2, 3, 4, 6, 8\}$ (c) $\{6, 8\}$ (d) ϕ
(e) $\{1, 2, 3, 4\}$ (f) $\{1, 3, 5, 7\}$ (g) $\{6\}$ (h) $\{5, 7\}$

5.20 $\forall x \in A \cap B$, x is in B. By definition of $A \setminus B$
$x \notin A \setminus B$ $\therefore$ $(A \cap B) \cap (A \setminus B) = \phi$ $\therefore$ $A \cap B$ and $A \setminus B$ disjoint.

To show $A = (A \cap B) \cup (A \setminus B)$.

"$\subseteq$": $\forall x \in A$, if $x \in B$, then $x \in A \cap B$ $\therefore x \in (A \cap B) \cup (A \setminus B)$
if $x \notin B$, then $x \in A \setminus B$. $\therefore x \in (A \cap B) \cup (A \setminus B)$

$\therefore$ $A \subseteq (A \cap B) \cup (A \setminus B)$

"$\supseteq$": $\forall x \in (A \cap B) \cup (A \setminus B)$.
$\therefore$ $x \in A \cap B$ or $x \in A \setminus B$
if $x \in A \cap B$, then $x \in A$
if $x \in A \setminus B$, then $x \in A$
$\therefore$ $A \supseteq (A \cap B) \cup (A \setminus B)$.

So, $A = (A \cap B) \cup (A \setminus B)$

5.21. True. Both are equal to $A \cap B$. Let's show $A \setminus (A \setminus B) = A \cap B$.
If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus $x \notin A \setminus B$
$\therefore$ $x \in A \setminus (A \setminus B)$ $\therefore$ $A \cap B \subseteq A \setminus (A \setminus B)$.
Conversely, if $x \in A \setminus (A \setminus B)$, then $x \in A$ and $x \notin A \setminus B$.
If $x \notin B$, then since $x \in A$, $x \in A \setminus B$ contradiction
$\therefore$ $x \in B$ $\therefore$ $x \in A \cap B$ $\therefore$ $A \setminus (A \setminus B) \subseteq A \cap B$

5.25. (b) $\bigcup_{n \in \mathbb{N}} (1, 1+\frac{1}{n}) = (1, 2)$. $\bigcap_{n \in \mathbb{N}} (1, 1+\frac{1}{n}) = \phi$

(c) $\bigcup_{\substack{x \in \mathbb{R} \\ x > 2}} [2, x] = [2, \infty)$, $\bigcap_{\substack{x \in \mathbb{R}, \\ x > 2}} [2, x] = \{2\}$

(d) $\bigcup_{B \in \mathcal{B}} B = [0, 5)$, $\bigcap_{B \in \mathcal{B}} B = [2, 3]$

- Mark "T" or "F".

(1) T (2) T (3) **F** (4) T (5) F

(example: $A = \{1,2\}$, $B = \{1,3\}$, $x = 3 \notin A \setminus B = \{2\}$)

6.10 (b) True. If $(x,y) \in (A \cup B) \times C$, then $x \in A \cup B$, $y \in C$. $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

If $x \in A$, then $(x,y) \in A \times C$

If $x \in B$, then $(x,y) \in B \times C$

$\therefore$ $(x,y) \in (A \times C) \cup (B \times C)$ $\therefore$ $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$

If $(x,y) \in (A \times C) \cup (B \times C)$, then $(x,y) \in A \times C$ or $(x,y) \in B \times C$

If $(x,y) \in A \times C$, then $x \in A$, $y \in C$. $\therefore$ $(x,y) \in (A \cup B) \times C$

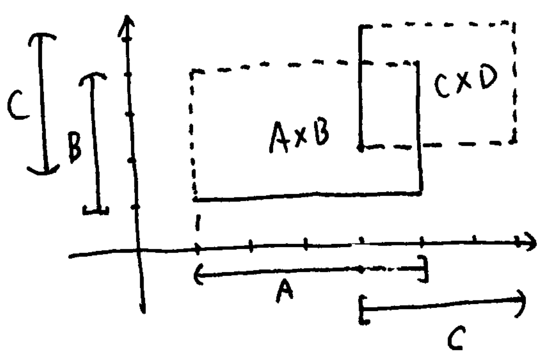
If $(x,y) \in B \times C$, then $x \in B$, $y \in C$ $\therefore$ $(x,y) \in (A \cup B) \times C$

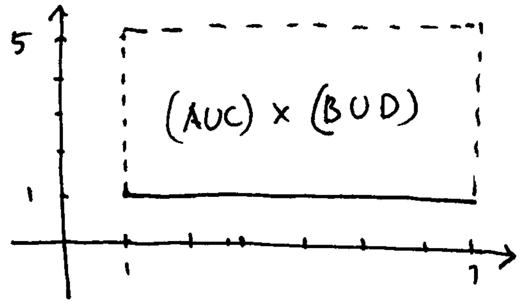
$\therefore$ $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$

(d) False. a counterexample:

$A = (1, 5]$, $B = [1, 4)$, $C = [4, 7)$, $D = (2, 5)$

$\therefore$ $A \cup C = (1, 7)$, $B \cup D = [1, 5)$





6.11 (f) Symmetric.

(g) reflexive, symmetric, transitive

for xRy iff $(x-y)^2 \geq 0$

Answer to the original statement in the ~~book~~

xRy iff $(x-y)^2 < 0$

is: symmetric and transitive.

The reason is: $(x-x)^2$ is NOT < 0 $\therefore$ NOT reflexive.

If xRy, this means $(x-y)^2 < 0$. but this is false

so from the truth table of "$p \Rightarrow q$", q is true

when p is false $\therefore$ If $\underbrace{xRy}_{p}$, then $\underbrace{yRx}_{q}$.

If $\underbrace{xRy \text{ and } yRz}_{p \text{ false}}$, then $\underbrace{xRz}_{q}$ true

6.14:

$(x,y) R (x,y)$: $x+y = y+x$ holds $\therefore$ reflexive

$(\underset{a}{x}, \underset{b}{y}) R (\underset{c}{x}, \underset{d}{y})$: $\underset{a}{x}+\underset{d}{y} = \underset{b}{y}+\underset{c}{x}$

If $(a,b) R (c,d)$, then $a+d = b+c$ $\therefore (c,d) R (a,b)$ symmetric

If $(a,b) R (c,d)$ and $(c,d) R (e,f)$

then $\begin{cases} a+d = b+c & (1) \\ c+f = d+e & (2) \end{cases}$ sum up (1) and (2):

$a+f = b+e$

$\because (a,b) R (e,f)$ $\therefore$ transitive.

$\therefore R$ is an equivalent relation

$$E_{(7,3)} = \{(x,y) : (x,y) R (7,3)\}$$

$$= \{(x,y) : x+3 = y+7\}$$

$\therefore E_{(7,3)}$ is the line $y = x-4$.

6.20. $x R y$ iff $x - y = 3k$, some $k \in \mathbb{Z}$

$x - x = 0 = 3 \cdot \underset{\substack{\uparrow \\ k}}{0}$ $\therefore$ $x R x$

If $x R y$, then $x - y = 3k$ $\therefore$ $y - x = 3(-k)$

since $-k \in \mathbb{Z}$ $\therefore$ $y R x$

If $x R y$ and $y \in z$, then $x - y = 3k_1$, $y - z = 3k_2$ for some $k_1, k_2 \in \mathbb{Z}$

$\therefore$ $x - z = 3(k_1 + k_2)$, since $k_1 + k_2 \in \mathbb{Z}$

$\therefore$ $x R z$ $\therefore$ R is an equivalent relation

$$E_5 = \{x : x R 5\} = \{x : x - 5 = 3k, \ k \in \mathbb{Z}\}$$
$$= \{3k + 5 : k \in \mathbb{Z}\}$$
$$= \{3k + 2 : k \in \mathbb{Z}\}.$$

There're 3 distinct equiv. classes.

$E_0 = \{3k : k \in \mathbb{Z}\}$

$E_1 = \{3k + 1 : k \in \mathbb{Z}\}$

$E_2 = \{3k + 2 : k \in \mathbb{Z}\}$

$E_3 = E_0$

$E_4 = E_1$ $\ldots$