Problem 1 (15 points). Determine whether the following series converge or diverge. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\frac{\frac{3^{n+1}}{(n+1)^3}}{\frac{3^n}{n^3}} = 3 \cdot \left(\frac{n}{n+1}\right)^3 \to 3 > 1$$

By Ratio Test, diverges

(b) $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)$

$$(-1)^n \left(1 + \frac{1}{n}\right) \not\to 0$$

$\therefore$ diverges. by the n^{th} term test.

(c) $\sum_{n=2}^{\infty} \frac{\sqrt{n} + \sin n}{n(\sqrt{n} - 1)}$

$$\frac{\sqrt{n} + \sin n}{n(\sqrt{n} - 1)} \geq \frac{\sqrt{n} - 1}{n(\sqrt{n} - 1)} = \frac{1}{n}$$

$\sum \frac{1}{n}$ diverges By comparison Test, diverges.

Problem 2 (15 points). Prove the following limits exist and then evaluate them.

(a) $\lim_{n\to\infty}\left(\sqrt{9n^6+6n^3}-3n^3\right)$

$$\sqrt{9n^6+6n^3}-3n^3 = \frac{(9n^6+6n^3)-9n^6}{\sqrt{9n^6+6n^3}+3n^3} = \frac{6n^3}{\sqrt{9n^6+6n^3}+3n^3}$$

$$= \frac{6}{\sqrt{9+\frac{6}{n^3}}+3} \longrightarrow \frac{6}{3+3} = 1.$$

(b) $\lim_{n\to\infty} x_n$, where $x_1 = 2, x_{n+1} = \sqrt{3x_n+4}$

$(x_n) \nearrow$: $x_2 = \sqrt{3\cdot 2+4} = \sqrt{10} > 2$

If $x_{n+1} \geq x_n$, then

$x_{n+2} = \sqrt{3x_{n+1}+4} \geqslant \sqrt{3x_n+4} = x_{n+1}$. induction concludes

(x_n) bdd above by 4.

$x_1 = 2 < 4$

If $x_n \leq 4$, then

$x_{n+1} = \sqrt{3x_n+4} \leq \sqrt{3\cdot 4+4} = 4$. induction concludes.

To find limit:

$x = \sqrt{3x+4}$ $\therefore$ $x^2-3x-4=0$.

$\therefore$ $x = -1$, or 4

$(x_n) \nearrow$ to 4, $\therefore$ $x = 4$.

Problem 3 (12 points) For each of the following statement, circle one answer and only one answer. You do not need to give reasons.

(a) The set of boundary points of the set $\bigcap_{n=1}^{\infty}(0, \frac{1}{n})$ is $\{0\}$.

Answer: True False

(b) If f and g are injective functions from $\mathbb{R}$ to $\mathbb{R}$, then $f \circ g$ and $g \circ f$ are both injective.

Answer: True False

(c) Let P be the statement: For all $x \in (0, 1)$, $1 < f(x) < 10$. The negation of P is: there exists $x \in (-\infty, 1] \cup [1, \infty)$, such that $f(x) \leq 1$ or $f(x) \geq 10$.

Answer: True False

(d) If $a_n > 0$ and the sequence (a_n) is unbounded, then $(\frac{1}{a_n})$ converges to 0.

Answer: True False

(e) If (a_n) and (b_n) are both divergent sequences of real numbers, then the sequence $(a_n b_n)$ is also divergent.

Answer: True False

(f) The sequence (a_n) is given by: $a_1 = 8, a_n = (-1)^n \frac{5n}{n+1}$ for $n \geq 2$. Then $\limsup a_n$ is

Answer: (A) 8 (B) 5 (C) -5 (D) Does not exist

Problem 4 (25 points). Mark each statement True or False. If True, give a proof. If False, give a counter-example.

(a) Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be functions. Then $f + g : \mathbb{N} \to \mathbb{N}$ is not surjective.

True. $f \geq 1$, $g \geq 1$. $f+g \geq 2$.

1 has no preimage.

(b) Let A and B be two sets and $A \subseteq B$. If A is denumerable and B is uncountable, then $B \backslash A$ is uncountable.

True. If $B \backslash A$ is countable.
then $B = (B \backslash A) \cup A$ countable
as A countable since it's denumerable
and union of two countable sets is countable

(c) Let x and y be two given real numbers. If $x < y\epsilon$ for any $\epsilon > 0$ and $y \geq 0$, then $x \leq 0$.

True. If $y = 0$. $x < 0$.
If $y > 0$. $\frac{x}{y} < \varepsilon$, $\forall \varepsilon > 0 \Rightarrow \frac{x}{y} \leq 0 \Rightarrow x \leq 0$.
as $y > 0$

(d) Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. Suppose that for any positive rational number $r \in \mathbb{Q}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $|a_n - a| < r$. Then $\lim_{n\to\infty} a_n = a$.

True $\forall \varepsilon > 0$, $\exists\, r > 0$, $r \in \mathbb{Q}$, s.t. $0 < r < \varepsilon$ by density of $\mathbb{Q}$

For $r > 0$, $\exists\, n_0$, $\forall n > n_0$. $|a_n - a| < r < \varepsilon$

$\therefore$ def. of limit $\Rightarrow$ $\lim\limits_{n\to\infty} a_n = a$.

(e) Let $S = (0,1) \cap \mathbb{Q}$, then S is neither open nor closed.

True not open: $\operatorname{int} S = \phi$, $S \neq \phi$.

not closed: $\operatorname{bd} S = [0,1]$, not contained in S

Problem 5 (8 points) Let A and B be subsets of the interval $(0, 1)$ with $\sup A = \sup B = 1$. Let $C = \{ab : a \in A, b \in B\}$. Prove that $\sup C = 1$.

$\forall a \in A \Rightarrow 0 < a < 1$, $\quad \therefore \quad 0 < ab < 1$
$\forall b \in B \Rightarrow 0 < b < 1$

$\therefore$ 1 is an upper bound of C.

$\forall \varepsilon > 0$, we show: $1-\varepsilon$ is not upper bdd of C.

We can assume $0 < \varepsilon < 1$. (Otherwise, $1-\varepsilon \le 0$ is not an upper bound of C

$\sup A = 1. \Rightarrow \exists a \in A$, s.t. $a > 1 - \frac{\varepsilon}{2} > 0$

$\sup B = 1 \Rightarrow \exists b \in B$. s.t. $b > 1 - \frac{\varepsilon}{2} > 0$

$\therefore ab > 1 - \varepsilon + \frac{\varepsilon^2}{4} > 1 - \varepsilon$ $\quad \therefore$ $1-\varepsilon$ is not an upper bound of C.

$\therefore$ 1 is the least upper bdd of C, hence

$\sup C = 1$.

Problem 6 (10 points) Let $f : A \to B$ be a function and let S and T be subsets of A. If $f(S) = f(T)$ and f is injective, show $S = T$.

$\forall s \in S$, $\because f(s) \in f(S) = f(T)$.

$\exists t \in T$. s.t. $f(s) = f(t)$

$\because f$ is injective, $\therefore s = t$ $\therefore s \in T$

$\therefore S \subseteq T$.

Similarly, $T \subseteq S$

$\therefore S = T$.

Problem 7 (15 points) Mark each statement True of False, if True, give a proof, if False, provide a counterexample.

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

False. $a_n = \frac{-1}{\sqrt{n}}$, alternating series test

$\Rightarrow$ Σa_n converge

But $\Sigma a_n = \Sigma \frac{1}{n}$ diverge.

(b) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

True

Pf1. $\sum_{n=1}^{\infty} |a_n|$ converge $\Rightarrow$ $\lim |a_n| = 0$.

$\therefore$ $\exists n_1$, $\forall n > n_1$, $|a_n| < 1$.

$\forall \varepsilon > 0$, $\exists n_2$, $\forall m, n > n_2$.

$||a_m| + \cdots + |a_n|| < \varepsilon$ Cauchy Criterion for Series

$\therefore$ $||a_m^2| + \cdots + |a_n^2|| < ||a_m| + \cdots + |a_n|| < \varepsilon$

Cauchy Criterion Series again

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n^2$ converge

Pf2. $\sum_{n=1}^{\infty} |a_n| < \infty$ $\Rightarrow$ $\lim |a_n| = 0$

$\Rightarrow$ $|a_n| < M$, $\forall n \in \mathbb{N}$ for some $M < +\infty$.

$\therefore$ $\sum_{n=1}^{\infty} a_n^2 \leq \sum_{n=1}^{\infty} M|a_n| \leq M \sum_{n=1}^{\infty} |a_n| < \infty$