SECOND MIDTERM EXAM

Math 220, Section 201, March 23, 2010

Name (Last, First): ____________________

Student Number: ____________________

There are four problems in this exam which add up to 40 points.

Problem 1. [12/40] For each S given below, find int S (the set of interior points of S), bd S (the set of boundary points of S), S' (the set of accumulation points of S), and answer whether S is a closed subset of $\mathbb{R}$. (**You do NOT need to justify your answers**)

(a) $S = [-1,3] \cup (3,4]$ $\quad S = [-1, 4]$

- int $S =$ $(-1, 4)$
- bd $S =$ $\{-1, 4\}$
- $S' =$ $[-1, 4]$
- Is S a closed subset in $\mathbb{R}$: Yes No.

(b) $S = (-1, \sqrt{2}) \cup (\sqrt{2}, 4)$

- int $S =$ $(-1, \sqrt{2}) \cup (\sqrt{2}, 4)$
- bd $S =$ $\{-1, \sqrt{2}, 4\}$
- $S' =$ $[-1, 4]$
- Is S a closed subset in $\mathbb{R}$: Yes No.

(c) $S = \bigcup_{n=1}^{\infty} \left[-\frac{1}{n}, 1 - \frac{1}{n}\right]$ $= [-1, 1)$

- int $S =$ $(-1, 1)$
- bd $S =$ $\{-1, 1\}$
- $S' =$ $[-1, 1]$
- Is S a closed subset in $\mathbb{R}$: Yes No.

Problem 2. [12/40] Mark each statement True or False. You do NOT need to justify your answers.

(1) Let S be a subset of $\mathbb{R}$. If $S \cap \mathrm{bd}S \neq \emptyset$, then S cannot be an open subset of $\mathbb{R}$.

True. False.

(2) For a subset S of $\mathbb{R}$, if both $\min S$ and $\max S$ exist, then S is closed.

True. False.

(3) If A and B are bounded subsets of $\mathbb{R}$, then $\sup(A \cup B)$ exists and is a real number.

True. False.

(4) Let $x, y \in \mathbb{R}$ such that $|x| < y + \epsilon$ for every $\epsilon > 0$. Then $y > 0$.

True. False.

(5) Let S be a subset of $\mathbb{R}$. If there are at least two distinct elements in $N(x;\epsilon) \cap S$ for every $\epsilon > 0$, then x is an accumulation point of S.

True False.

(6) If S is countable, then its closure $cl\, S$ is also countable.

True. False.

Problem 3. [8/40] Use mathematical induction to prove: for all $n \in \mathbb{N}$

$$\frac{(2n)!}{2^{2n-1}(n!)^2} \geq \frac{1}{n}.$$

1). $n = 1$. $\frac{(2\cdot 1)!}{2^{2\cdot 1-1}(1!)^2} = \frac{2}{2\cdot 1} = 1 \geqslant \frac{1}{1}$ holds.

2) Assume $n = k$, inequality holds

When $n = k+1$,

$$\frac{\big(2(k+1)\big)!}{2^{2(k+1)-1}\left[(k+1)!\right]^2} = \frac{(2k)!\,(2k+1)(2k+2)}{2^{2k-1}\cdot 2^2\cdot [k!]^2(k+1)^2}$$

$$= \frac{(2k)!}{2^{2k-1}(k!)^2}\cdot\frac{2k+1}{2\cdot(k+1)}$$

$$\geq \frac{1}{k}\cdot\frac{2k+1}{2}\cdot\frac{1}{k+1}$$

$$\geq \frac{1}{k+1}.$$

$\therefore$ Math. Induction $\Rightarrow$ $\frac{(2n)!}{2^{2n-1}(n!)^2} \geqslant \frac{1}{n}$, $\forall n \in \mathbb{N}$

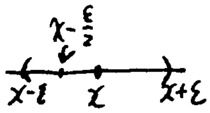
Problem 4. [8/40] Let S be a subset of $\mathbb{R}$ and assume $\inf S$ exists. Prove that $\inf S \in \operatorname{bd} S$.

Set $x = \inf S$.

Claim: $\forall \varepsilon > 0$, $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \phi$. ①

Otherwise $\exists\, N(x;\varepsilon) \subseteq S$

$\Rightarrow x - \frac{\varepsilon}{2} \in S$ But $x - \frac{\varepsilon}{2} < x$,



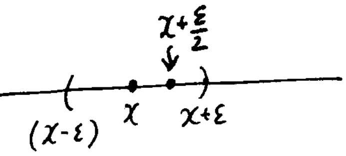
Contradicts to x is a lower bound of S.

Claim: $\forall \varepsilon > 0$, $N(x;\varepsilon) \cap S \neq \phi$ ②

Otherwise, $\exists\, N(x;\varepsilon) \subseteq \mathbb{R} \setminus S$

$\Rightarrow x + \frac{\varepsilon}{2} < s$, $\forall s \in S$.

Contradicts to x is the greatest lower bound of S



$\therefore x \in \operatorname{bd} S$.