Solution HW 9

18.2. (a) False. Example: $a_n = (-1)^n/n$. (b) True. This is the part (a) of Theorem 18.8. (c) False. The statement is: a sequence of rational numbers is convergent in $\mathbb{Q}$ if and only if it is Cauchy. Counterexample: $\sqrt{2}$ can be represented as the limit of (s_n) where s_n is the nth decimal approximation of $\sqrt{2}$, namely, $s_1 = 1.4, s_2 = 1.41, s_2 = 1.414, ...$

18.3. (c) Claim 1: $s_n \le 3$. We use induction to show this. $s_1 = 1 < 3$. If $s_k \le 3$, then $s_{k+1} = \frac{1}{4}(2s_k + 5) \le \frac{1}{4}(2 \cdot 3 + 5) \le 3$. Hence the claim 1 is true by induction. Claim 2: $s_{n+1} \ge s_n$. We prove this by induction. $s_2 = 7/4 > 1 = s_1$. If $s_{k+1} \ge s_k$, then $s_{k+2} = \frac{1}{4}(2s_{k+1} + 5) \ge \frac{1}{4}(2s_k + 5) = s_{k+1}$. So Claim 2 holds by induction. Therefore (s_n) is an increasing sequence which is bounded above, and the monotone convergence theorem implies (s_n) converges to some $s \in \mathbb{R}$. To find s, we solve $s = \frac{1}{4}(2s + 5)$ which implies $s = 5/2$.

(e) Claim 1: s_n is increasing. $s_2 = \sqrt{13} > 3 = s_1$. If $s_{k+1} \ge s_k$ then $s_{k+2} = \sqrt{10s_{k+1} - 17} \ge \sqrt{10s_k - 17} = s_{k+1}$. By induction, claim 1 holds. Claim 2: $s_n \le 10$. Again, we use induction. $s_1 = 3 < 10$. If $s_k \le 10$, then $s_{k+1} = \sqrt{10s_k - 17} \le \sqrt{10s_k} \le \sqrt{10 \cdot 10} = 10$. So claim 2 is true by induction. The monotone convergence theorem then implies (s_n) converges to some $s \in \mathbb{R}$. To find s, we observe $s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{10s_n - 17} = \sqrt{10s - 17}$. Squaring both sides, we solve $s^2 - 10s + 17 = 0$ and note $s \ge 0$ because $s_n \ge 0$ and we get $s = 5 + 2\sqrt{2}$.

18.4. (a) $a_n = (-1)^n/n$. The sequence (a_n) is Cauchy (because it is convergent) and not monotone (neither increasing nor decresing). (b) $a_n = n$. The sequence (a_n) is monotone but not Cauchy (because it diverges). (c) $a_n = (-1)^n$. The sequence (a_n) is bounded but not Cauchy (the sequence diverges).

18.5. (a) False. Counterexample: take $a_n = 1/n, b_n = -1/n^2$. $c_1 = a_1 + b_1 = 0, c_2 = 1/2 - 1/4 = 1/4, c_3 = 1/3 - 1/9 = 2/9 < 1/4$. $c_1 < c_2$ so (c_n) not decreasing; $c_2 > c_3$ so (c_n) not increasing.

(b) Falese. Counterexample: $a_n = 1/n, b_n = 1 - 1/n$, $c_n = a_n b_n = 1/n - 1/n^2$, this is the same c_n in part (a), hence not monotone.

18.7. Claim 1: $s_n \le 6$. $s_1 = \sqrt{6} < 6$. If $s_k \le 6$ then $s_{k+1} = \sqrt{6 + s_k} \le \sqrt{6+6} < 6$. By induction we see claim 1 holds. Claim 2: $s_{n+1} \ge s_n$. $s_2 = \sqrt{6 + \sqrt{6}} > \sqrt{6} = s_1$. If $s_{k+1} \ge s_k$ then $s_{k+2} = \sqrt{6 + s_{k+1}} \ge \sqrt{6 + s_k} = s_{k+1}$, so induction guarantees claim 2 is true. Now the monotone convergence theorem ensures the (s_n) converges to some real number s. Further, we have $s = \sqrt{6 + s}$,, which implies $s = 3$.

18.10. (a) Since $|r| < 1$, $\lim_{n\to\infty} r^{n+1} = 0$. Hence

$$\lim_{n\to\infty} (1 + r + ... + r^n) = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}.$$

(b)

$$0.9999... = \frac{9}{10} \lim_{n\to\infty} \left(1 + \frac{1}{10} + ... + \frac{1}{10^n}\right) = \frac{9}{10} \cdot \frac{1}{1 - 1/10} = 1.$$