Solution to HW 8

16.2. (a) True because $s_n \le |s_n|$ or because $|s_n| < \varepsilon \Leftrightarrow -\varepsilon < s_n < \varepsilon$.

(b) False: let $s_n = -1$ for all $n \in \mathbb{N}$.

(c) False: Theorem 16.8 requires that $a_n \to 0$. A counterexample is $s_n = (-1)^n$, $s = 0$, $k = 1$ and $a_n = 1$ for $n \in \mathbb{N}$.

(d) True: Theorem 16.14.

16.6 (a) Given $\varepsilon > 0$, let $N = |k|/\varepsilon$.

(b) Given $\varepsilon > 0$, let $N = (1/\varepsilon)^{1/k}$. Then $n > N$ implies $n^k > 1/\varepsilon$, so $(1/n^k) < \varepsilon$.

(c) $\left|\frac{3n+1}{n+2} - 3\right| = \left|\frac{-5}{n+2}\right| \le \frac{5}{n}$, so given $\varepsilon > 0$, let $N = 5/\varepsilon$.

(d) $\left|\frac{\sin n}{n} - 0\right| \le \frac{1}{n}$, so given $\varepsilon > 0$, let $N = 1/\varepsilon$.

(e) $\left|\frac{n+2}{n^2-3}\right| \le \frac{2n}{\frac{1}{2}n^2} = \frac{4}{n}$ when $n \ge 3$. Now $n^2 - 3 \ge \frac{1}{2}n^2$ iff $\frac{1}{2}n^2 \ge 3$ iff $n^2 \ge 6$ iff $n \ge 3$.

So given $\varepsilon > 0$, let $N = \max\{4/\varepsilon, 3\}$.

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(b) $\left|\frac{4n^2-7}{2n^3-5}\right| \le \frac{4n^2}{2n^3-5} \le \frac{4n^2}{n^3} = \frac{4}{n} = 4\left(\frac{1}{n}\right)$ for $n \ge 2$.

(d) Hint in book: $\sqrt{n}/(n+1) < \sqrt{n}/n = 1/\sqrt{n}$.

(e) $\frac{n^2}{n!} = \frac{n \cdot n}{n(n-1)(n-2)\cdots(2)(1)} \le \frac{n}{n^2-3n+2} \le \frac{n}{n^2-3n} = \frac{1}{n-3} \le \frac{1}{\frac{n}{2}} = \frac{2}{n}$ when $n \ge 6$.

The inequalities above and Th. 16.8 imply that (b), (d) and (e) hold.

16.8 (a) $(a_n) = 2n$ is not bounded. Hence, (a_n) is divergent by Th. 16.13.

(b) Suppose that $(b_n) = (-1)^n$ is convergent. Then $\lim b_n = \lim (-1)^n = s$ for some $s \in \mathbb{R}$.

For $\varepsilon = 1$, $\exists N \in \mathbb{R}$ s.t.

$$|s - (-1)^n| < 1 \quad \text{for } n > N. \qquad (1)$$

Let n be even and $n > N$. By (1), we have

$$|s-1| = |s-(-1)^n| < 1 \Rightarrow -1 < s-1 < 1 \Rightarrow s > 0 \qquad (2)$$

Let n be odd and $n > N$. By (1), we have

$$|s+1| = |s-(-1)^n| < 1 \Rightarrow -1 < s+1 < 1 \Rightarrow s < 0 \qquad (3)$$

Hence, we get a contradiction by (2) and (3)

This proves that $(b_n) = (-1)^n$ is divergent.

16.12 (a) Since (t_n) is bounded, $\exists M > 0 \ni |t_n| \le M, \forall n$. Given $\varepsilon > 0$, $\exists N \ni |s_n| < \varepsilon/M$ whenever $n > N$. Thus for $n > N$ we have $|s_n t_n| = |s_n||t_n| < |s_n| M < (\varepsilon/M)(M) = \varepsilon$, so $s_n t_n \to 0$.

(b) Let $s_n = 1/n$ and $t_n = n$, so that $s_n t_n = 1$ for all n. Then $\lim s_n t_n = 1 \neq 0 = \lim s_n$.

16.14 Since $s > 0$, $\varepsilon = s/2 > 0$. Thus $\exists N \ni n > N$ implies $|s_n - s| < \varepsilon/2$. But then $0 < s/2 < s_n, \forall n > N$.

17.5 (b) $|s_n| = \left|\frac{(-1)^n}{n+3}\right| = \frac{1}{n+3}$ and $\lim \frac{1}{n+3} = 0$. Hence, $\lim s_n = 0$.

(d) $s_n = \frac{2^{3n}}{3^{2n}} = \left(\frac{2^3}{3^2}\right)^n = \left(\frac{8}{9}\right)^n \to 0$ as $n \to \infty$.

(f) $\frac{3+n-n^2}{1+2n} \underset{(n\geq 3)}{\leq} \frac{3+n-n^2}{3n} = \frac{1}{n} + \frac{1}{3} - \frac{1}{3}n$.

Since $\lim\left(\frac{1}{n}+\frac{1}{3}-\frac{1}{3}n\right) = -\infty$, $\lim \frac{3+n-n^2}{1+2n} = -\infty$

by Th. 17.12 (b).

(k) $s_n = \frac{n^2}{2^n} \Rightarrow \frac{s_{n+1}}{s_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \left(1+\frac{1}{n}\right)\frac{1}{2} \to \frac{1}{2}$ as $n \to \infty$.

By Th. 17.7, $\lim s_n = 0$.

17.15 (a) Hint in book: Multiply and divide by $\sqrt{n+1}+\sqrt{n}$.

Answer: $\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}} \to 0$

(b) $\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \leq \frac{1}{n} \to 0$

(c) $\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1} \to \frac{1}{2}$

17.16 Let $t_n = 1/s_n$. Then $\lim (t_{n+1}/t_n) = \lim (s_n/s_{n+1}) = 1/L$. If $L > 1$, then $1/L < 1$, so $\lim t_n = 0$. Thus $\lim s_n = +\infty$ by Theorem 17.13.

17.17 (a) Hint in book: For $k > 0$, use the ratio test.

Answer: Let $s_n = \frac{k^n}{n!}$. Then for $k > 0$, $\frac{s_{n+1}}{s_n} = \frac{k^{n+1}}{k^n} \cdot \frac{n!}{(n+1)!} = \frac{k}{n+1} \to 0$. Thus $s_n \to 0$.

For $k < 0$ we have

$$\left|\frac{k^n}{n!} - 0\right| = \frac{|k|^n}{n!}, \text{ so } \frac{k^n}{n!} \to 0 \text{ by Theorem 16.8.}$$

When $k = 0$, $s_n = 0 \ \forall n$, so $s_n \to 0$.

(b) When $k > 0$, $\lim n!/n^k = +\infty$ (diverges to $+\infty$) by Theorem 17.13. When $k = 0$, the sequence is not defined. When $k < 0$, it diverges with no limit.