Solution to HW set 6.

10.3 For $n=1$, $1^2 = \frac{1}{6} 1(1+1)(2\cdot 1+1)$. Hence, the equation is true for $n=1$. Assume that the equation is true for $n=k$, i.e. $1^2+2^2+\cdots+k^2 = \frac{1}{6}k(k+1)(2k+1)$.

Then

$$1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$

$$= \frac{1}{6}(k+1)\left(k(2k+1)+6(k+1)\right) = \frac{1}{6}(k+1)(2k^2+7k+6)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1),$$

which proves that the equation is also true for $n=k+1$. By induction, the equation is true for all $n \in \mathbb{N}$. //

10.8 Here is the key induction step:

$$\frac{n}{2n+1} + \frac{1}{4(n+1)^2-1} = \frac{n}{2n+1} + \frac{1}{(2n+2)^2-1} = \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{n(2n+3)+1}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}. \quad //$$

10.11 Here is the key induction step:

$$1-\frac{1}{(n+1)!}+\frac{n+1}{(n+2)!}=1-\frac{(n+2)-(n+1)}{(n+2)!}=1-\frac{1}{(n+2)!}.$$

10.18 Here is the key induction step.

$$\frac{(2n)!}{n!}(4n+2)=\frac{(2n)!\,2(2n+1)(n+1)}{n!\,(n+1)}=\frac{(2n)!\,(2n+1)(2n+2)}{(n+1)!}=\frac{(2n+2)!}{(n+1)!}$$

10.21 (a) True for $n = 1$ and $n \geq 4$. For $n = 1$ we have $1^2 \leq 1!$, which is true. For $n = 4$ we have $4^2 = 16 \leq 24 = 4!$, which is also true. Now suppose that $k^2 \leq k!$ for some $k \geq 4$. Then $k+1 = k(1+1/k) < k^2$, since $1 + 1/k \leq 2 < k$. Thus $(k+1)^2 = (k+1)(k+1) < (k+1)(k^2) \leq (k+1)(k!) = (k+1)!$. It follows from Theorem 10.6 that $n^2 < n!$ for all $n \geq 4$.

(b) True for all $n \in \mathbb{N}$ except $n = 3$. Verify $n = 1$ and $n = 2$ separately. Then use induction on $n \geq 4$. Note that $2k+1 \leq 2k+k = 3k \leq k^2$ when $k \geq 3$. So if $k^2 \leq 2^k$ then

$$(k+1)^2 = k^2 + 2k + 1 \leq k^2 + k^2 = 2k^2 \leq 2(2^k) = 2^{k+1}.$$

(c) True for all $n \geq 4$. Indeed, $2^4 = 16 \leq 24 = 4!$, so it holds for $n = 4$. Now suppose $2^k \leq k!$ for some $k \geq 4$. Then

$$2^{k+1} = 2(2^k) \leq 2(k!) \leq (k+1)(k!) = (k+1)!.$$

It follows from Theorem 10.6 that $2^n \leq n!$ $\forall\, n \geq 4$.

12.3 (d) $\sup(0,4)=4$, $\max(0,4)$ does not exist.

(f) $\sup\{1-\frac{1}{n} : n\in\mathbb{N}\}=1$, $\max\{1-\frac{1}{n} : n\in\mathbb{N}\}$ does not exist.

(h) $\sup\{(-1)^n(1+\frac{1}{n}) : n\in\mathbb{N}\}=\frac{3}{2}=\max\{(-1)^n(1+\frac{1}{n}) : n\in\mathbb{N}\}$

(n) $\sup\{r\in\mathbb{Q} : r^2<5\}=\sqrt{5}$, $\max\{r\in\mathbb{Q} : r^2<5\}$ does not exist.

12.4 (d) $\inf\,[0,4) = 0$, $\min[0,4)$ does not exist.

(f) $\inf\{1-\frac{1}{n} : n\in\mathbb{N}\} = \min\{1-\frac{1}{n} : n \in \mathbb{N}\} = 0$.

(h) $\inf\{(-1)^n(1+\frac{1}{n}) : n\in\mathbb{N}\} = \min\{(-1)^n(1+\frac{1}{n}) : n\in\mathbb{N}\} = -2$.

(n) $\inf\{r\in\mathbb{Q} : r^2<5\} = -\sqrt{5}$, $\min\{r\in\mathbb{Q} : r^2<5\}$ does not exist.

12.5 Suppose $m \in S$. Since $m = \sup S$, m is an upper bound for S. Hence $m = \max S$. Conversely, if $m = \max S$, then $m \in S$ by the definition of maximum.

12.7 (a) If $k = 0$, then the result is trivial. So suppose $k > 0$. Let $m = \sup S$. Since m is an upper bound for S, km is an upper bound for kS. Now given any $\varepsilon > 0$, since $m = \sup S$, $\exists\, s \in S \ni m - \varepsilon/k < s$. But then $km - \varepsilon < ks$, so that $km - \varepsilon$ is not an upper bound for kS. Thus km is the least upper bound for kS.

The proof that $\inf(kS) = k \cdot \inf S$ is similar.

(b) Suppose $k < 0$ and let $m = \inf S$. Then $m \le s\ \forall\, s \in S$. Thus $km \ge ks\ \forall\, s \in S$ and km is an upper bound for kS. Now given $\varepsilon > 0$, $\exists\, s \in S \ni m + (\frac{\varepsilon}{-k}) < s$. But then $km - \varepsilon < ks$ so that $km - \varepsilon$ is not an upper bound for kS. Thus km is the least upper bound for kS.

The proof that $\inf(kS) = k \cdot \sup S$ is similar.

12.12 (a) Let $m = \sup f(D)$ and $n = \sup g(D)$. Then $\forall\, x \in D$,

$$(f+g)(x) = f(x) + g(x) \le m + n.$$

Thus $m + n$ is an upper bound for $(f+g)(D)$. It follows that the *least* upper bound for $(f+g)(D)$ is also less than or equal to $m + n$. That is, $\sup[(f+g)(D)] \le m + n$.

(b) Let $D = [0, 1]$, $f(x) = x$, and $g(x) = 1 - x$. Then $f(D) = g(D) = [0, 1]$, and $\sup f(D) = \sup g(D) = 1$. But $(f+g)(D) = \{1\}$, so that $\sup(f+g)(D) = 1 < 2 = \sup f(D) + \sup g(D)$.

(c) $\inf[(f+g)(D)] \ge \inf f(D) + \inf g(D)$. The proof is similar to part (a).

12.13 Hint in the book: Let $S = \{q \in \mathbb{Q} : q < x\}$. Then S is bounded above by x and we can let $y = \sup S$. Prove that $y = x$ by showing that $y < x$ and $y > x$ both lead to contradictions.

Proof: Let $S = \{q \in \mathbb{Q} : q < x\}$. Then S is bounded above by x and we can let $y = \sup S$. We will prove $y = x$ by showing $y < x$ and $x < y$ are not possible.

Suppose $y < x$. Then by the density Theorem 12.12, $\exists\, q_0 \in \mathbb{Q} \ni y < q_0 < x$. This contradicts y being an upper bound for S.

Suppose $x < y$. Then by Theorem 12.12 again, $\exists\, q_1 \in \mathbb{Q} \ni x < q_1 < y$. But then $\forall$ rational $q < x$ we have $q < x < q_1$. This implies q_1 is an upper bound of S that is smaller than y, a contradiction.