Solution to HW5

8.1 (a) True. Definition 8.1

(b) False. S might be ϕ.

(c) False. It (the cardinal number) is transfinite

(d) False. The domain of the bijection should be $\mathbb{N}$

(e) True. Theorem 8.9.

(f) False. Counterexample: $\{1,2\} \subseteq \mathbb{N}$. $\{1,2\}$ is finite

8.2 (a) False. $f: \mathbb{N} \to S$ must be surjective.

counterexample: $S = \mathbb{R}$. $f: \mathbb{N} \to \mathbb{R}$, $f(n) = n$ is injective. But $\mathbb{R}$ uncountable

(b) True. $\mathbb{Q}$ is countable, not finite.

(c) True. Theorem 8.10.

8.3 (a) $f(x) = 2x+1$ is one possibility

(d) $f(x) = \dfrac{x}{1-x}$ or $f(x) = \dfrac{1}{x} - 1$

8.4. (a) $f(x) = (n-m)x + m$

(b) Given two intervals (a,b), (c,d). by (a),

$\exists$ bijective $f: (0,1) \to (a,b)$

$\exists$ bijective $g: (0,1) \to (c,d)$

$\therefore$ $g \circ f^{-1}: (a,b) \to (c,d)$ is bijective

$\therefore$ (a,b) and (c,d) are equinumerous.

8.10. S is denumerable $\Rightarrow$ $\exists$ bijective $f: \mathbb{N} \to S$.

Define $g: \mathbb{N} \to S \setminus \{f(1)\}$ — a proper subset of S

by $g(n) = f(n+1)$

g is injective: If $g(n_1) = g(n_2)$, then $f(n_1+1) = f(n_2+1)$

So $n_1+1 = n_2+1$ since f is injective

$\therefore$ $n_1 = n_2$

g is surjective: $\forall\, s \in S \setminus \{f(1)\}$, $s = f(n)$ for some $n > 1$, $n \in \mathbb{N}$ since f is surjective.

$\therefore$ $g(n-1) = s$ $\therefore$ g is bijective $\therefore$ $S \setminus \{f(1)\}$ denumerable