Solution to HW set 4.

(Math 220)

6.23. (a) $\cdot\ a^2 = a^2 \Rightarrow (a,a)R(a,a)$.

$\cdot\ (a,b)R(c,d) \Rightarrow ab = cd \Rightarrow cd = ab \Rightarrow (c,d)R(a,b)$.

$\cdot\ (a,b)R(c,d)$ and $(c,d)R(h,k) \Rightarrow ab = cd$ and $cd = hk$

$\Rightarrow ab = hk \Rightarrow (a,b)R(h,k)$.

Hence, R is an equivalence relation on A.

(b) $(a,b) \in E_{(9,2)} \Leftrightarrow ab = 9\cdot 2 = 18 \Leftrightarrow a$ is a divisor of 18 and $b = \frac{18}{a}$. Hence, we get

$$E_{(9,2)} = \{(9,2),(2,9),(6,3),(3,6),(1,18),(18,1)\}.$$

(c) $E_{(1,2)} = \{(1,2),(2,1)\}$. In general,

$E_{(1,p)} = \{(1,p),(p,1)\}$, where p is a prime.

(f) $E_{(9,2)}$ is the hyperbola $xy = 18$.

7.2 (a) True: Definition 7.5.
(b) False: f must be bijective. If f is not 1–1, then $f^{-1}(y) \subseteq A$.
(c) False: if f is not surjective, then $f^{-1}(D)$ may be empty.

For example, If $A = B = \{1,2\}$, $f(1) = f(2) = 1$ and $D = \{2\}$, then $f^{-1}(D) = \phi$.

(d) True: Theorem 7.18.
(e) True. See the comment after Definition 7.22.
(f) False. The identity function maps $\mathbb{R}$ onto $\mathbb{R}$ by $i(x) = x$ for all $x \in \mathbb{R}$.

7.7 (c) f is not injective because $f(0)=f(1)$.

f is surjective. In fact, $\forall a \in \mathbb{R}$, $x^3-x-a \to \infty$ as $x\to\infty$ and $x^3-x-a\to-\infty$ as $x\to-\infty$. Hence, the equation $x^3-x-a=0$ always has a solution in $\mathbb{R}$ (by the property of continuous function in Calculus), which implies that there exists $x\in\mathbb{R}$ such that $f(x)=a$. So f is surjective.

(d) f is bijective because $f(1)=0$, $\lim_{x\to\infty} f(x)=\infty$ and f is a continuous increasing function.

7.8 (a) f is surjective. In fact, $\forall a\in[0,\infty)$, let C be a circle with the radius $\sqrt{\frac{a}{\pi}}$. then $f(C)=a$.

f is not injective. In fact, let C_i be the circle whose equation is

$$(x-i)^2+y^2=1, \quad \text{where } i=1,2.$$

Clearly $C_1\neq C_2$. However, $f(C_1)=\pi=f(C_2)$.

(b) Both injective and surjective.

7.13 (a) $\{b, c, e\}$ (b) $\{1, 2, 3\}$.

7.20 (a) True. If $f(C) \subseteq D$, then

$x \in C \Rightarrow f(x) \in f(C) \subseteq D \Rightarrow x \in f^{-1}(D)$. Hence, $C \subseteq f^{-1}(D)$

Conversely, if $C \subseteq f^{-1}(D)$, then

$x \in f(C) \Rightarrow x = f(a)$ for some $a \in C \subseteq f^{-1}(D)$

$\Rightarrow x = f(a) \in f(f^{-1}(D)) \subseteq D$. Thus, $f(C) \subseteq D$

(b) f is bijective. In fact, let f be bijective.

We have

$f(C) = D$ implies that $\forall x \in C$, $f(x) \in f(C) = D$. or $x \in f^{-1}(D)$.

This proves that $C \subseteq f^{-1}(D) = f^{-1}(f(C)) \underset{\uparrow}{=} C \Rightarrow C = f^{-1}(D)$.

Th. 7.17 (a) (See Remark ①. below);

Also, we have

$C = f^{-1}(D) \Rightarrow f(C) = f(f^{-1}(D)) \underset{\uparrow}{=} D$

Th. 7.17 (b) (see Remark ② below).

Remark ①: f is injective $\Rightarrow f^{-1}(f(C)) = C$. (1)

Proof 1). $\forall x \in f^{-1}(f(C))$, $f(x) \in f(C)$ $\therefore \exists a \in C$

s.t. $f(x) = f(a)$. f is injective $\Rightarrow x = a \in C$

Thus, $f^{-1}(f(C)) \subseteq C$. (2)

2) $\forall a \in C$, $f(a) \in f(C)$. $\therefore a \in f^{-1}(f(C))$. Thus

$C \subseteq f^{-1}(f(C))$. (3)

③

By (2) and (3), (1) holds.

Remark ②: f is surjective $\Rightarrow f(f^{-1}(D)) = D$. (4)

Proof 1) $\forall y \in f(f^{-1}(D))$, $\exists x \in f^{-1}(D)$ s.t. $y = f(x)$.

Since $x \in f^{-1}(D) \Rightarrow f(x) \in D$, $y = f(x) \in D$. Thus,

$$f(f^{-1}(D)) \subseteq D \qquad (5)$$

2) $\forall y \in D$, f is surjective $\Rightarrow \exists a \in A$ s.t. $y = f(a)$

$\therefore a \in f^{-1}(D)$. Hence, $y = f(a) \in f(f^{-1}(D))$. Thus,

$$D \subseteq f(f^{-1}(D)) \qquad (6)$$

By (5) and (6), (4) holds.

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(b) If $y \in f(A)\setminus f(C)$, then $y \in f(A)$ and $y \notin f(C)$. Since $y \in f(A)$, $\exists\, x \in A \ni f(x) = y$. Since $y \notin f(C)$, $x \notin C$. Thus $x \in A\setminus C$, and so $y = f(x) \in f(A\setminus C)$.

(c) Hint in book: If f is injective, then equality holds.
Proof: Let $y \in f(A\setminus C)$. Since f is injective $\exists$ a unique x in $A \ni f(x) = y$. Now suppose y were in $f(C)$. Then $x \in C$, since x is the only point in A that maps onto y. But $y \in f(A\setminus C)$, so we must also have $x \in A\setminus C$; whence $x \notin C$. This contradiction means that $y \notin f(C)$. Now $x \in A$, so $y = f(x) \in f(A)$. Thus $y \in f(A)\setminus f(C)$ and $f(A\setminus C) \subseteq f(A)\setminus f(C)$. The reverse inclusion follows from part (b).

7.26 Let $A = [0,\infty)$, $B = \mathbb{R}$, and $C = [0,\infty)$. Define $f(x) = x$ and $g(x) = x^2$. Then $g \circ f$ is injective since dom $g \circ f = [0,\infty)$, but g is not injective since dom $g = \mathbb{R}$.