Solutions to HW 10

§19.2 (a) False: let $s_n = n$.

(b) True: Th. 19.7.

(c) False: only true if $s \in \mathbb{R}$.

(d) True: See Fig. 19.2 on Page 185. or Th. 19.11 (b)

(e) False: $(s_n) = (1, 0, 2, 0, 3, 0, \cdots)$ Then $\limsup s_n = +\infty$ and $\liminf = 0$.

19.3 Answers in book:

(a) $S = \{-1, 1\}$, $\limsup s_n = 1$, $\liminf s_n = -1$.

(b) $S = \{0\}$, $\limsup t_n = \liminf t_n = 0$.
(c) $S = \{-\infty, 0\}$, $\limsup u_n = 0$, $\liminf u_n = -\infty$.
(d) $S = \{-\infty, 0, +\infty\}$, $\limsup v_n = +\infty$, $\liminf v_n = -\infty$.

19.5 Hints in book: the value of each limit is given, but not the derivation.

(a) This is a subsequence of $\left(1+\frac{1}{n}\right)^n$, so the limit is e.

(b) $\left[\left(1+\frac{1}{n}\right)^n\right]^2 \to e^2$

(c) $\left(1+\frac{1}{n}\right)^{n-1} = \left(1+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^n \to 1 \cdot e = e$

(d) $\left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1+\frac{1}{n}\right)^{-n} = \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} \to e^{-1}$

(e) $\left(1+\frac{1}{2n}\right)^n = \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}} \to \sqrt{e}$ by part (a).

(f) Let $k = n+1$. Then $s_{k-1} = \left(\frac{k+1}{k}\right)^{k+2} = \left(1+\frac{1}{k}\right)^2\left(1+\frac{1}{k}\right)^k \to 1^2 \cdot e = e$.

19.6 (a) False. Let $s_n = (-1)^n n$.
(b) True. If a sequence oscillates, then its limit inferior and limit superior are unequal. It follows that it cannot converge, for if it converged all its subsequences would converge to the same limit.
(c) False. Let $s_n = n$.

19.7 (a) True. Theorem 19.7 says a bounded sequence has a convergent subsequence, and Theorem 18.12 says that the convergent subsequence is Cauchy.
(b) False. Let $s_n = n$.

19.11 Hint in book: Use Exercise 16.15(b).

Proof: Let (t_n) be a convergent sequence in S. By Exercise 16.15(b), it suffices to show that $t = \lim t_n$ is in S. Since (s_n) is bounded, S is bounded, so $t \in \mathbb{R}$.

Since some subsequence of (s_n) converges to t_1, $\exists\, n_1 \ni |s_{n_1} - t_1| < 1$. Likewise, $\exists\, n_2 > n_1 \ni |s_{n_2} - t_2| < \frac{1}{2}$. In general, choose $n_k > n_{k-1} \ni |s_{n_k} - t_k| < \frac{1}{k}$. It follows that

$$|s_{n_k} - t| \le |s_{n_k} - t_k| + |t_k - t| < \frac{1}{k} + |t_k - t|.$$

Thus $0 \le \lim |s_{n_k} - t| \le \lim \frac{1}{k} + \lim |t_k - t| = 0 + 0 = 0$. Hence t is the limit of the subsequence (s_{n_k}), and so $t \in S$.

19.13 (a) Hint in book: Use Theorem 19.11.

Proof: Let $\limsup s_n = s$ and $\limsup t_n = t$. By Theorem 19.11(a), given any $\varepsilon > 0\ \exists\, N_1 \ni n > N_1$ implies that $s_n < s + \varepsilon/2$. Likewise, $\exists\, N_2 \ni n > N_2$ implies that $t_n < t + \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then $n > N$ implies that $s_n + t_n < (s + t) + \varepsilon$. So given $\varepsilon > 0$, there are only finitely many $n \ni s_n + t_n \ge (s + t) + \varepsilon$. Thus no subsequence of $(s_n + t_n)$ can converge to anything greater than $(s + t)$. That is, $\limsup (s_n + t_n) \le s + t$.

(b) Let $(s_n) = (1, 0, 1, 0, \ldots)$ and $(t_n) = (0, 1, 0, 1, \ldots)$.

32.4 (a) $\lim (-1)^n$ does not exist $\Rightarrow \sum (-1)^n$ is divergent

(b) $\lim \frac{n}{n+1} = 1 \ne 0 \Rightarrow \sum \frac{n}{n+1}$ is divergent.

(c) $\lim \frac{n}{\sqrt{n^2+1}} = 1 \ne 0 \Rightarrow \sum \frac{n}{\sqrt{n^2+1}}$ is divergent.

(d) $\lim \cos\frac{n\pi}{2}$ does not exist $\Rightarrow \sum \cos\frac{n\pi}{2}$ is divergent.

32.5

(a) $\frac{1}{1-\frac{1}{3}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

(b) $\frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} = 2 - 1\frac{3}{4} = \frac{1}{4}$

(f) A telescoping series with terms $\left(\frac{1/2}{2n-1} - \frac{1/2}{2n+1}\right)$. The sum is the first term: 1/2.

(h) $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$. We have $\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) \to \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$.

(i) A telescoping series with terms $\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$. The sum is the first term: 1/2.

(k) $\frac{1}{n(n+1)(n+2)} = \frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}$, so this is a telescoping series. The sum is the first term: 1/4.

(l) $\frac{1}{n(n+1)\cdots(n+k)} = \frac{1/k}{n(n+1)\cdots(n+k-1)} - \frac{1/k}{(n+1)(n+2)\cdots(n+k)}$, so this is a telescoping series. The sum is the first term: $\frac{1/k}{1(2)(3)\cdots(1+k-1)} = \frac{1}{k(k!)}$.

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32.8 Since $a_n \geq 0 \ \forall\ n$, the sequence (s_n) of partial sums is increasing. By the Monotone Convergence Theorem 18.3, (s_n) is convergent iff it is bounded.

32.9 Hint in book: Rationalize the denominator and look at the partial sums.
Solution: $a_n = \sqrt{n+1} - \sqrt{n}$ and $s_n = \sqrt{n+1} - 1$. Hence the series diverges to $+\infty$.

32.10 (a) Let s_n by the nth partial sum of $\sum y_n$. Then

$$\begin{aligned} s_n &= \sum_{k=1}^{n} y_k = y_1 + y_2 + \cdots + y_n \\ &= (x_1 - x_2) + (x_2 - x_3) + \cdots + (x_n - x_{n+1}) \\ &= x_1 - x_{n+1}\,. \end{aligned}$$

Thus (s_n) converges iff (x_n) converges. That is $\sum y_n$ converges iff (x_n) converges.
(b) If $\sum y_n$ converges, let $L = \lim x_n$. Then $\sum y_n = x_1 - L$.

32.11 Hint in book: Use Theorem 32.6.
Solution: Suppose that (b_n) is bounded by $M > 0$ so that $|b_n| \leq M$ for all n. Given $\varepsilon > 0$, Theorem 32.6 implies $\exists\ N \ni$ if $n \geq m > N$, then

$$\big||a_m| + |a_{m+1}| + \cdots + |a_n|\big| < \varepsilon/M.$$

Then if $n \geq m > N$ we have

$$\begin{aligned} |a_m b_m + a_{m+1} b_{m+1} + \cdots + a_n b_n| &\leq M\big||a_m| + |a_{m+1}| + \cdots + |a_n|\big| \\ &< M(\varepsilon/M) = \varepsilon. \end{aligned}$$

Thus $\sum a_n b_n$ is convergent by Theorem 32.6.

33.3 (a) converges because $\lim \frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n}\right)^3 \frac{1}{2} = \frac{1}{2} < 1$.

(b) converges because $\lim \frac{a_{n+1}}{a_n} = \lim \frac{2}{n+1} = 0 < 1$.

(e) Converges because $\lim \frac{a_{n+1}}{a_n} = \lim \frac{n^n}{(n+1)^n} = \frac{1}{e} < 1$.

(g) Diverges because $\frac{1}{\sqrt{n(n+1)}} \geq \frac{1}{\sqrt{(n+1)^2}} = \frac{1}{n+1} \geq \frac{1}{2n}$ and

$\sum \frac{1}{2n} = +\infty$.

(m) Converges because $\lim |a_n|^{\frac{1}{n}} = \lim (2^n e^{-n})^{\frac{1}{n}} = \frac{2}{e} < 1$.

(n) Diverges because $\lim |a_n|^{\frac{1}{n}} = \lim (3^n e^{-n})^{\frac{1}{n}} = \frac{3}{e} > 1$.

(p) Converges because $\lim \frac{a_{n+1}}{a_n} = \lim \frac{n+1}{2n+3} = \frac{1}{2} < 1$

33.4 (b) Let $f(x) = \frac{1}{x(\ln x)(\ln\ln x)^p}$. $f'(x) < 0$ for $x \geq 3$.

Hence, $f(x)$ is positive and decreasing. Thus, Integral Test can be used. If $p \neq 1$, $\int_3^n \frac{dx}{x(\ln x)(\ln\ln x)^p} \underset{u=\ln\ln x}{=\!=} \int_{\ln\ln 3}^{\ln\ln n} \frac{du}{u^p} =$

$= \frac{1}{1-p}\left((\ln\ln n)^{1-p} - (\ln\ln 3)^{1-p}\right)$. It follows that

$\lim_{n\to\infty}\left(\int_3^n \frac{dx}{x(\ln x)(\ln\ln x)^p}\right)$ will be finite if $p>1$ and infinite if $p<1$. Thus, $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)^p}$ converges if $p>1$ and diverges if $p<1$.

When $p=1$, $\lim_{n\to\infty} \int_3^n \frac{dx}{x(\ln x)(\ln\ln x)^p} =$

$= \lim_{n\to\infty}(\ln\ln\ln n - \ln\ln\ln 3) = +\infty$. Hence, the series is also divergent when $p=1$.

By the facts above, $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)^p}$ converges iff $p>1$.

33.5 (a) converges conditionally because $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by Th. 33.16, and $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right|$ diverges by Th. 33.13.

(b) diverges by Th. 33.8 (b).

(c) converges absolutely Th. 33.8 (a)

(f) converges conditionally because $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by Th. 33.16, and $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right|$ diverges by Th. 33.13.

(i) $\left(\frac{\ln n}{n} \right)_{n=2}^{\infty}$ is a decreasing sequence of positive numbers. Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by Th. 33.16. Also, $\sum_{n=2}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right|$ diverges by Th. 33.13. Therefore, $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges conditionally. //

33.7 Hint in book: Look at the sequence (b_n/a_n).

Solution: Since $0 < \frac{b_{n+1}}{a_{n+1}} \le \frac{b_n}{a_n}$ $\forall\, n$, the sequence $\left(\frac{b_n}{a_n}\right)$ is decreasing. In particular, $\frac{b_n}{a_n} \le \frac{b_1}{a_1}$ $\forall\, n$.

Thus $0 < b_n \le \frac{b_1}{a_1} a_n$ $\forall\, n$, so Σb_n converges by the Comparison Test with the convergent series $\Sigma \frac{b_1}{a_1} a_n$.

33.9 Hint in book: Consider the series $\Sigma(a_{n+1} - a_n)$.
Solution: The series $\sum |a_{n+1} - a_n|$ converges by comparison with $\sum b_n$, so $\Sigma(a_{n+1} - a_n)$ also converges by Theorem 33.5. Let $A = \Sigma(a_{n+1} - a_n)$. Then

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} (a_{k+1} - a_k) \\ &= \lim_{n\to\infty} [(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n)] \\ &= \lim_{n\to\infty} [a_{n+1} - a_1] \\ &= \left(\lim_{n\to\infty} a_{n+1}\right) - a_1. \end{aligned}$$

Thus $\lim a_n = A + a_1$ and (a_n) is convergent.